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Solar Radiation Pressure and the Motion of Earth Satellites

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The main works devoted to the study of the effects of solar radiation pressure on the motion of artificial earth satellites are reviewed. The resonance case, when the motion of the satellite undergoes long-period variations with large amplitudes, is considered in detail. The graphs given in the paper permit easy determination of the presence of resonance for a wide range of satellite orbits, provided that the orbital elements are known. The graphs were constructed on the assumption that the effect of the earth's shadow can be neglected.

Introduction

ONLY a few years ago, when the intense work of creating a theory of artificial satellite motion was begun, it was assumed that the effect of light pressure or of solar radiation is negligibly small as compared to the influence of the earth's oblateness and the effects of lunar-solar gravitation. However, the investigations of the motion of the satellite "Vanguard 1" (1958 β_2) showed that, even if the total lunar-solar gravitation effect is taken into account, no full agreement between theory and observation is obtained. It was precisely the need of explaining this discrepancy which prompted the study of the effect of light pressure, thus initiating the theoretical and experimental work in this field. Various authors (Musen,[12] Parkinson, Jones, and Shapiro[15]) analyzed the orbital perturbations of the satellite 1958 β_2 due to radiation pressure, and the secular perturbations were determined in the first approximation. It was assumed that the orbit is continuoulsy illuminated and that the reflection from the satellite surface is specular. The effect of re-radiation from the earth and the Poynting-Robertson accelerating effect were disregarded. The magnitude of the acceleration due to radiation pressure, estimated for the satellite 1958 β_2, was found to be approximately 10^{-5} cm/sec^2.

By processing the observations of the satellite 1958 β_2 over a period of approximately two years from launching, a perigee-height perturbation with an amplitude of 1–2 km and a period of about 850 days was determined. In conjunction with the lunar-solar effect, which yields roughly the same amplitude but only half the period, allowance for this perturbation results in good agreement with theory. Yet, in this case, the observed effect of radiation pressure was so small that even toward the end of the two-year period the magnitude of the effect was still determined with an accuracy of not more than 30%. The accuracy of determining the magnitude of the radiation pressure from observations of the satellite "Echo 1" (1960 ι_1) was much greater, even during the first month of observations, since the corresponding variations in the orbit of "Echo 1" are much greater than in the orbit of "Vanguard 1." These variations in the orbit of

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"Echo 1" were determined with a relatively high accuracy (up to 3%); they showed that for satellites of large "area-to-mass" ratio the long-period terms due to radiation pressure alter the orbital parameters considerably, and hence alter the satellite lifetime.

At first glance, it seems unlikely that the effects of radiation pressure and of solar attraction could be comparable; yet the principal part of the effect of solar gravitational vanishes, since the earth experiences the same acceleration under this effect as a satellite. The net effect of the geocentric force of solar attraction during one complete revolution is almost zero, whereas, in the case of the radiation-pressure effect, the resultant acceleration of the satellite is considerably higher than the corresponding acceleration of the earth. Therefore, the radiation-pressure effect has to be taken into account in processing the observations of artificial satellites, in those cases in which the effect is significant.

1. Light Pressure

The mechanical action of light, exerted on reflecting and absorbing bodies, is called the pressure of light. It was discovered experimentally for the first time and measured in 1899 by the distinguished Russian physicist P. N. Lebedev.[1]

Theoretically, the pressure of light can be interpreted on the basis of the electromagnetic, as well as the quantum, theory of light. According to electromagnetic theory, the pressure exerted on a surface by a plane electromagnetic wave which falls perpendicularly on it is equal to the density of the electromagnetic energy (i.e., the energy per unit volume) near the surface. This energy consists of the energy of both incident and reflected waves. If the power of the incident electromagnetic wave per unit surface is S(erg/cm²sec), and the power of the corresponding reflected wave is $S\mathfrak{R}$(erg/cm²-sec), where $\mathfrak{R}$ denotes the reflection coefficient, then the energy density of the electromagnetic wave near the surface is $S[(1 + \mathfrak{R}/c)]$, where c is the speed of light. The light pressure on the surface of a body is precisely equal to this quantity

$$\mathfrak{P}r = \frac{S(1 +)\mathfrak{R}}{c}\left[\frac{\text{force}}{(\text{length})}\right] \tag{1}$$

From the viewpoint of quantum theory, light is a flow of particles—photons, each of which has an energy $h\nu$ and momentum $h\nu/c$, where h is Planck's constant and ν the frequency of the light. The number of photons which fall on 1 cm² of a surface, perpendicular to the ray, in 1 sec, is $N = S/h\nu$, and the momentum imparted by them to 1 cm² of surface in 1 sec is $N(h\nu/c) = S/c$.

If the surface absorbs all the photons, then the light pressure is S/c. If the surface reflects light partially, then the number of reflected photons which carry momentum of opposite sense is $N' = \mathfrak{R}(S/h\nu)$, as a result of which the surface acquires an additional momentum; the total momentum per second, i.e., the light pressure, is again $\mathfrak{P}r = [S(1 + \mathfrak{R})/c]$. In the general case, the force of light pressure per unit surface is

$$\mathfrak{P}_r = \frac{S(1 + \mathfrak{R})}{c}\cos^2\alpha \tag{2}$$

where α is the angle of incidence of light on the surface.

The light pressure is very small even for such a powerful radiation source as the sun. The power of solar radiation which falls on 1 cm² of earth surface per unit of time, or the so-called "solar constant," is

$$S_0 \approx [2.00 \pm 0.04]\,\frac{\text{cal}}{\text{cm}^2\,\text{min}} \approx 1.35\cdot 10^6\,\frac{\text{erg}}{\text{cm}^2\,\text{sec}} \tag{2a}$$

For a celestial body which is at a distance r from the sun, the solar constant for this distance is determined by the formula

$$S = S_0(r_0/r)^2 \tag{3}$$

where r_0 is the mean distance earth-sun.

The value of the reflection coefficient R depends on the reflecting properties of the surface and varies from 0 to 1. For an absolute black body $R = 0$, for a specular surface $R = 1$. Substituting (3) in (2), we finally obtain

$$\mathfrak{P}_r = \frac{S_0(1 + \mathfrak{R})}{c}\left(\frac{r_0}{r}\right)^2\cos^2\alpha \tag{4}$$

in particular, for a sphere with an ideal specular surface that is situated on the earth's orbit:

$$\mathfrak{P}_0 = \frac{2S_0}{c} = 0.90\cdot 10^{-4}\,\frac{\text{dyne}}{\text{cm}^2}\left[\frac{\text{g}}{\text{cm sec}^2}\right] \tag{5}$$

In the foregoing units, we have the following values of $\mathfrak{P}_0$ at planetary orbits:

Mercury	$0.50\cdot 10^{-3}$
Venus	$1.81\cdot 10^{-4}$
Earth	$0.90\cdot 10^{-4}$
Mars	$0.45\cdot 10^{-5}$
Jupiter	$0.10\cdot 10^{-7}$

Let us now consider the acceleration acting on the body. Suppose a body of mass m which is situated in cosmic space, is subjected to the gravitational force of the sun and to the solar radiation pressure force. These forces act in precisely opposite directions. We have the following values for the acceleration of a body, due to solar gravitation, at planetary orbits:

Venus	$1.2\cdot 10^{-3}g$
Earth	$6.2\cdot 10^{-4}g$
Mars	$2.7\cdot 10^{-4}g$
Jupiter	$1.5\cdot 10^{-5}g$

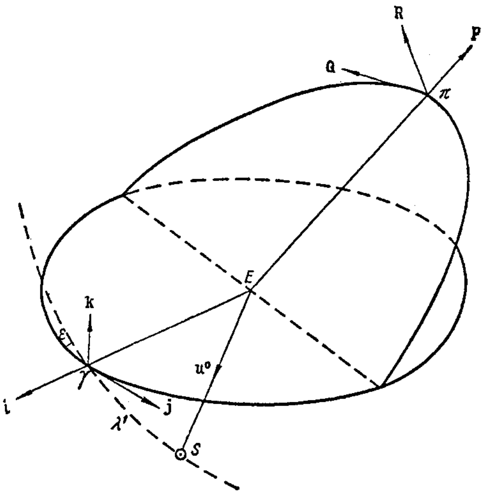
where $g = 980.62$ (cm/sec²) is the acceleration of the earth's gravity at the latitude of 45°. On the earth's orbit, the sun's gravitational force is 6.10^{-1} (cm/sec²).

The acceleration experienced by a body of mass m and effective cross section A under the action of radiation pressure, is determined by the formula

$$F = \mathfrak{P}_r\frac{A}{m}\left[\frac{\text{length}}{(\text{time})^2}\right] \tag{6}$$

The quantity A/m is a parameter which is constant for each body, and is called the "area-to-mass" ratio. This parameter is of decisive importance in estimating the effect of radiation pressure on the motion of artificial satellites, which is evident from the following approximate calculation: for the satellite "Vanguard 1" ($m = 1457$ g, $A = 207.6$ cm²) we have $A/m = 0.1425$ (cm²/g) and $F = 1.28\cdot 10^{-5}$ (cm/sec²), whereas for the satellite "Echo 1" ($m_0 = 71215$ g, $A_0 = 7292901$ cm²) we obtain initially (immediately after launching) the value $A_0/m_0 = 102$(cm²/g), which yields $F = 0.92\cdot 10^{-2}$(cm/sec²), i.e., a value roughly 100 times larger. By comparing the numerical data it becomes evident that for artificial satellites the effect of solar radiation is still small as compared to the gravitational effect of the sun, yet the relationship between these effects can substantially change for particular A/m. Indeed, if a light satellite (for example a gas-inflated balloon of type "Echo 1") is of spherical shape, then the sun's gravitational force is proportional to the volume, i.e., to the cube of the sphere's radius ρ, whereas the radiation-pressure force is proportional to the effective cross section, i.e., to the square of the radius. These forces act in opposite directions and their ratio is proportional to $1/\rho$. Hence, in the case of a light sphere of sufficiently small size, both forces can balance each other or the radiation-pressure force can exceed the gravitational force of the sun. In the latter case, the body will leave

Fig. 1.



the solar system under the effect of radiation pressure. This makes it feasible to employ radiation pressure for future space flights, using a "solar sail" which creates a small, steady, thrust. Such motion would take place along a helical orbit and a change in the orientation of the sail with respect to the solar radiation from which it acquires its power would permit not only motion toward the outer but also the inner planets, i.e., toward the sun.

2. Perturbations of Satellite Motion Due to Radiation Pressure

The theory of perturbations of satellite motion, due to radiation pressure, was first developed by Musen (1960a), who obtained an analytic expansion for long-period perturbations of orbital elements, while disregarding the shadow effect. Since the earth's shadow causes changes only in perturbation amplitude, without altering the nature of the perturbations, it is, in the general case, entirely justifiable to neglect the shadow effect in the first approximation. Musen deduced an analytic theory of resonance for the case in which the perigee of the orbit follows the sun; he showed that if the resonance conditions are satisfied, then the light pressure causes considerable variations in the orbit over a time interval of several months. In this case, long-period terms appear in the formulas for all the elements except the semimajor axis; it is precisely these terms which are of greatest interest.

The perturbations of orbital elements of a satellite, caused by the radiation-pressure effect, can be determined by the method of variation of vector elements (Musen[11,12]). Here we shall set forth, in brief, Musen's derivation, retaining as far as possible his notation.

Suppose **i**, **j**, and **k** are unit vectors with origin at the vernal equinox γ, related by the formula $\mathbf{j} = \mathbf{k} \times \mathbf{i}$, where **i** is in a radial direction from γ and **k** is normal to the earth's equatorial plane (Fig. 1).

P, **Q**, and **R** are unit vectors with origin at the perigee of the satellite's orbit π, related by the formula $\mathbf{Q} = \mathbf{R} \times \mathbf{P}$, where **P** is in a radial direction from the perigee and **R** is normal to the orbital plane; λ' is the mean longitude of the sun in the ecliptic, n' is the mean solar motion, ϵ is the inclination of the equator to the ecliptic, $\mathbf{u}^0$ is the unit vector of the direction earth-sun, i.e., the line of action of solar radiation with origin at the point E, $\mathfrak{M}$ is the mass of the earth and G is the gravitational constant.

Neglecting the eccentricity of the earth's orbit, we have for $\mathbf{u}^0$ the formula

$$\mathbf{u}_0 = \mathbf{i} \cos\lambda' + \mathbf{j} \sin\lambda' \cos\epsilon + \mathbf{k} \sin\lambda' \sin \epsilon \tag{7}$$

The effect of the solar radiation-pressure is expressed by the formula

$$\mathbf{F} = F\mathbf{u}^0 \tag{8}$$

where F is the acceleration of the satellite under the action of the radiation pressure which is constant for each spherical satellite and which is calculated by formula (6). In our case, F must be negative, since the direction of u^0 is opposite to the direction of the solar rays. In order to determine the perturbations in the orbital plane, Musen considers the perturbations of the vector **P**, introducing for this purpose the vector element $\mathbf{g} = e\mathbf{P}$. In a coordinate system fixed in the orbital plane, the motion of **P** is determined by the formula

$$G\mathfrak{M}(dg/dt) = \Gamma F \tag{9}$$

where

$$\Gamma = \frac{na^2}{\sqrt{1 - e^2}}\left\{-\frac{r}{a}\left(e \sin v + \frac{1}{2}\sin 2v\right)\mathbf{P}\cdot\mathbf{P} + \frac{1}{2}\frac{r}{a} \times \sin 2v\mathbf{Q}\cdot\mathbf{Q} + \left[\frac{r}{a}\left(\frac{1}{2} + e \cos v + \frac{1}{2}\cos 2v\right) + 1 - e^2\right]\mathbf{P}\cdot\mathbf{Q} - \left[\frac{1}{2}\frac{r}{a}(1 - \cos 2v) + 1 - e^2\right]\mathbf{Q}\cdot\mathbf{P}\right\} \tag{10}$$

Since the constant parts of the coefficients of $\mathbf{P}\cdot\mathbf{Q}$ and $\mathbf{Q}\cdot\mathbf{P}$ are equal to $\pm\frac{3}{2}(1 - e^2)$, and the constant parts of all the other coefficients are zero, it follows that for the long-period term of Γ we have

$$[\Gamma] = \tfrac{3}{2}na^2\sqrt{1 - e^2}(\mathbf{P}\cdot\mathbf{Q} - \mathbf{Q}\cdot\mathbf{P}) \tag{11}$$

Substituting (11) in (9), we obtain the long-period term of **P**:

$$\frac{d(e\mathbf{P})}{dt} = -\frac{3}{2}\frac{na^2\sqrt{1 - e^2}}{G\mathfrak{M}}\mathbf{R} \times \mathbf{F} \tag{12}$$

whence follows that $[d(e\mathbf{P})]/dt$ is normal to the component of the perturbing-acceleration vector **F** which lies in the orbital plane. Since the motion of **P** in a moving system of coordinates involves only rotation about **R** with angular velocity $\mathbf{R}(d\pi/dt)$, which alters the longitude of the perigee, it follows that

$$\frac{d\mathbf{P}}{dt} = \frac{d\pi}{dt}\mathbf{R} \times \mathbf{P} = \frac{d\pi}{dt}\mathbf{Q} \tag{13}$$

From (12) and (13) it follows that

$$\frac{de}{dt} = +\frac{3}{2}\frac{na^2\sqrt{1 - e^2}}{G\mathfrak{M}}\mathbf{Q}\cdot\mathbf{F}$$

$$e\frac{d\pi}{dt} = -\frac{3}{2}\frac{na^2\sqrt{1 - e^2}}{G\mathfrak{M}}\mathbf{P}\cdot\mathbf{F} \tag{14}$$

Substituting in (14) the expressions for **P** and **Q**:

$$\mathbf{P} = \mathbf{i}\left[+\cos^2\frac{i}{2}\cos(\omega + \Omega) + \sin^2\frac{i}{2}\cos(\omega - \Omega)\right] + \mathbf{j}\left[+\cos^2\frac{i}{2}\sin(\omega + \Omega) - \sin^2\frac{i}{2}\sin(\omega - \Omega)\right] + \mathbf{k}\sin i \sin\omega$$

$$\mathbf{Q} = \mathbf{i}\left[-\cos^2\frac{i}{2}\sin(\omega + \Omega) - \sin^2\frac{i}{2}\sin(\omega - \Omega)\right] - \mathbf{j}\left[+\cos^2\frac{i}{2}\cos(\omega + \Omega) - \sin^2\frac{i}{2}\cos(\omega - \Omega)\right] + \mathbf{k}\sin i \cos\omega \tag{15}$$

and recalling formulas (7) and (8), we obtain the formulas for the derivatives of the osculating elements which determine the satellite in the orbital plane:

$$\frac{de}{dt} = -\frac{3}{2}\frac{Fna^2\sqrt{1-e^2}}{G\mathfrak{M}} \times \Big[+\cos^2\frac{i}{2}\sin^2\frac{\epsilon}{2}\sin(\omega+\Omega+\lambda') + \sin^2\frac{i}{2}\times \sin^2\frac{\epsilon}{2}\sin(\omega-\Omega-\lambda') + \cos^2\frac{i}{2}\cos^2\times \frac{\epsilon}{2}\sin(\omega+\Omega-\lambda') + \sin^2\frac{i}{2}\cos^2\frac{\epsilon}{2}\sin(\omega-\Omega+ \lambda') - \frac{1}{2}\sin i\sin\epsilon\sin(\omega+\lambda') + \frac{1}{2}\sin i\sin\epsilon\sin(\omega-\lambda')\Big]$$

$$e\frac{d\pi}{dt} = -\frac{3}{2}\frac{Fna^2\sqrt{1-e^2}}{G\mathfrak{M}}\times \Big[+\cos^2\frac{i}{2}\sin^2\frac{\epsilon}{2}\cos(\omega+\Omega+ \lambda') + \sin^2\frac{i}{2}\sin^2\frac{\epsilon}{2}\cos(\omega-\Omega-\lambda') + \cos^2\frac{i}{2}\cos^2\frac{\epsilon}{2}\cos(\omega+\Omega-\lambda') + \sin^2\frac{i}{2}\cos^2\frac{\epsilon}{2}\cos(\omega-\Omega+\lambda') - \frac{1}{2}\sin i\sin\epsilon\cos(\omega+\lambda') + \frac{1}{2}\sin i\sin\epsilon\cos(\omega-\lambda')\Big] \tag{16}$$

Let us now consider the perturbations of elements that determine the position of the satellite in the orbital plane, i.e., the variation of **R** brought about only by the rotation of the osculating-orbit plane. The angular velocity of the osculating-orbit plane, regarded as a solid, is expressed by the formula

$$\psi = \frac{na^2}{\sqrt{1-e^2}}\frac{r}{a}\frac{\mathbf{R}\cdot\mathbf{F}}{G\mathfrak{M}} \tag{17}$$

Since the constant term of r is $-\frac{3}{2}\mathbf{P}ae$, it follows that the long-period term of ψ has the form

$$[\psi] = -\frac{3}{2}\frac{na^2e}{\sqrt{1-e^2}}\frac{\mathbf{P}\cdot\mathbf{R}\cdot\mathbf{F}}{G\mathfrak{M}} \tag{17a}$$

and the long-period term of **R** is

$$\frac{d\mathbf{R}}{dt} = +\frac{3}{2}\frac{na^2e}{\sqrt{1-e^2}}\mathbf{Q}\cdot\mathbf{R}\cdot\mathbf{F} \tag{18}$$

From (18), it is evident that the long-period term $d\mathbf{R}/dt$, which appears as a result of the radiation pressure, consists of rotation of **R** about **P** with an angular velocity which is proportional to the cosine of the angle between **R** and **F**. If **R** is perpendicular to **F**, i.e., to $\mathbf{u}^0$, then evidently $d\mathbf{R}/dt = 0$.

In order to find the equations for i and Ω, we shall consider the angular velocity $[\psi]$ as the geometric sum of the angular velocity di/dt of the orbital plane about the line of nodes, and the angular velocity $\sin i(d\Omega/dt)$ of rotation about the vector $\mathbf{P}\sin\omega + \mathbf{Q}\cos\omega$, whence we obtain for the sum of the long-period perturbations of i and Ω:

$$(\mathbf{P}\cos\omega - \mathbf{Q}\sin\omega)\frac{di}{dt} + (\mathbf{P}\sin\omega + \mathbf{Q}\cos\omega)\sin i\frac{d\Omega}{dt} = -\frac{3}{2}\frac{na^2e}{\sqrt{1-e^2}}\frac{\mathbf{P}\cdot\mathbf{R}\cdot\mathbf{F}}{G\mathfrak{M}} \tag{18a}$$

Multiplying successively by $(\mathbf{P}\cos\omega - \mathbf{Q}\sin\omega)$ and $(\mathbf{P}\sin\omega + \mathbf{Q}\cos\omega)$ and recalling that $(\mathbf{P}\cos\omega - \mathbf{Q}\sin\omega)\mathbf{k} = 0$ and $\mathbf{P}\cdot\mathbf{k} = \sin i\sin\omega$, we finally obtain

$$\frac{di}{dt} = -\frac{3}{2}\frac{na^2e\mathbf{R}\cdot\mathbf{F}\cos\omega}{G\mathfrak{M}\sqrt{1-e^2}}$$
$$\sin i\frac{d\Omega}{dt} = -\frac{3}{2}\frac{na^2e\mathbf{R}\cdot\mathbf{F}\sin\omega}{G\mathfrak{M}\sqrt{1-e^2}} \tag{19}$$

Setting $\mathbf{R} = \mathbf{l}\sin i\sin\Omega - \mathbf{j}\sin i\cos\Omega + \mathbf{k}\cos i$, and recalling (7) and (8), we obtain after some transformations

$$\frac{di}{dt} = -\frac{3}{4}\frac{Fna^2e}{G\mathfrak{M}\sqrt{1-e^2}}\Big[+\sin i\sin^2\frac{\epsilon}{2}\sin(\omega+\Omega+\lambda') - \sin i\sin^2\frac{\epsilon}{2}\sin(\omega-\Omega-\lambda') + \sin i\cos^2\frac{\epsilon}{2}\sin(\omega+\Omega- \lambda') - \sin i\cos^2\frac{\epsilon}{2}\sin(\omega-\Omega+\lambda') + \cos i\sin\epsilon\sin(\omega+\lambda') - \cos i\sin\epsilon\sin(\omega-\lambda')\Big] \tag{20}$$

$$\sin i\cdot\frac{d\Omega}{dt} = +\frac{3}{4}\frac{Fna^2e}{G\mathfrak{M}\sqrt{1-e^2}}\Big[+\sin i\sin^2\frac{\epsilon}{2}\cos(\omega+\Omega+ \lambda') - \sin i\sin^2\frac{\epsilon}{2}\cos(\omega-\Omega-\lambda') + \sin i\cos^2\frac{\epsilon}{2}\cos(\omega+ \Omega-\lambda') - \sin i\cos^2\frac{\epsilon}{2}\cos(\omega-\Omega+\lambda') + \cos i\sin\epsilon\cos(\omega+ \lambda') - \cos i\sin\epsilon\cos(\omega-\lambda')\Big]$$

The differential equations (16) and (20) can also be obtained by a different method, by directly using the differential equations for the osculating elements whose right-hand sides contain the components of the perturbing acceleration, as was done in Kozai's paper[8]:

$$\frac{da}{dt} = \frac{2na^3}{G\mathfrak{M}\sqrt{1-e^2}}F\left[Se\sin v + T\frac{p}{r}\right]$$
$$\frac{de}{dt} = \frac{na^2\sqrt{1-e^2}}{G\mathfrak{M}}F\left\{S\sin v + T\left[\cos v + \frac{1}{e}\left(1-\frac{r}{a}\right)\right]\right\}$$
$$\frac{di}{dt} = \frac{na^2}{G\mathfrak{M}\sqrt{1-e^2}}FW\frac{r}{a}\cos(v+\omega)$$
$$\sin i\frac{d\Omega}{dt} = \frac{na^2}{G\mathfrak{M}\sqrt{1-e^2}}FW\frac{r}{a}\sin(v+\omega) \tag{21}$$
$$\frac{d\omega}{dt} = -\cos i\frac{d\Omega}{dt} + \frac{na^2\sqrt{1-e^2}}{G\mathfrak{M}e}F\left[-S\cos v + T\left(1+\frac{r}{p}\right)\sin v\right]$$
$$\frac{dM}{dt} = n - \frac{2a^2}{G\mathfrak{M}}FS\frac{r}{a} - \sqrt{1-e^2}\left(\frac{d\omega}{dt} + \cos i\frac{d\Omega}{dt}\right)$$

where $\omega = \pi - \Omega$; FS, FT, and FW are the three components of the perturbing acceleration in a radial direction, in a direction perpendicular to the radial in the orbital plane, and in a direction perpendicular to the orbital plane, respectively; S, T, and W are determined by the formulas

$$S(v) = -\cos^2\frac{i}{2}\sin^2\frac{\epsilon}{2}\cos(u+\Omega+\lambda') - \sin^2\frac{i}{2}\sin^2\frac{\epsilon}{2}\cos(u-\Omega-\lambda') - \cos^2\frac{i}{2}\cos^2\frac{\epsilon}{2}\times \cos(u+\Omega-\lambda') - \sin^2\frac{i}{2}\cos^2\frac{\epsilon}{2}\cos(u-\Omega+\lambda') + \frac{1}{2}\sin i\sin\epsilon\cos(u+\lambda') - \frac{1}{2}\sin i\sin\epsilon\cos(u-\lambda') \tag{22}$$

$$T(v) = + \cos^2\frac{i}{2}\sin^2\frac{\epsilon}{2}\sin(u+\Omega+\lambda') + \sin^2\frac{i}{2}\sin^2\frac{\epsilon}{2}\sin(u-\Omega-\lambda') + \cos^2\frac{i}{2}\cos^2\frac{\epsilon}{2}\times \sin(u+\Omega-\lambda') + \sin^2\frac{i}{2}\cos^2\frac{\epsilon}{2}\sin(u-\Omega+\lambda') - \frac{1}{2}\sin i\sin\epsilon\sin(u+\lambda') + \frac{1}{2}\sin i\sin\epsilon\sin(u-\lambda')$$

$$W = -\sin i\cos^2\frac{\epsilon}{2}\sin(\Omega-\lambda') - \sin i\sin^2\frac{\epsilon}{2}\sin(\Omega+\lambda') - \cos i\sin\epsilon\sin\lambda'$$

If the quantities $S(v)$ and $T(v)$ in formulas (22) are expressed in the form

$$\begin{aligned} S(v) &= S(0)\cos v + T(0)\sin v \\ T(v) &= T(0)\cos v - S(0)\sin v \end{aligned} \qquad (22a)$$

and then substituted in (21), one again obtains, after averaging over the unperturbed mean anomaly M, i.e., after separating the long-period term, the differential equations (16) and (20). To these formulas, one can also add the differential equations for the perigee distance $q = a(1-e)$:

$$\frac{dq}{dt} = \frac{da}{dt} - a\frac{de}{dt} \qquad (22b)$$

Since the radiation pressure does not cause long-period perturbations in a, which is subjected to only small short-period perturbations and therefore varies but weakly, we shall adopt for the derivative of the perigee distance the formula

$$\frac{dq}{dt} = -a\frac{de}{dt} \qquad (23)$$

Thus, we obtained expressions for the derivatives of the osculating elements, in the form of a sum of trigonometric terms whose arguments are combinations of the angles λ', ω, and Ω, i.e., in the form of series

$$\frac{d\varkappa}{dt} = \pm A_{1\,\cos}^{\;\sin}(\omega+\Omega+\lambda') \pm A_{2\,\cos}^{\;\sin}(\omega-\Omega-\lambda') \pm A_{3\,\cos}^{\;\sin}(\omega-\Omega+\lambda') \pm A_{4\,\cos}^{\;\sin}(\omega\Omega+\lambda') \pm A_{5\,\cos}^{\;\sin}(\omega+\lambda') \pm A_{6\,\cos}^{\;\sin}(\omega-\lambda') \qquad (24)$$

where $\varkappa$ is an arbitrary element, and the coefficients $A_1, \ldots, A_6$ depend on the size and shape of the orbit and its inclination to the ecliptic. Assuming that $\dot\omega + \dot\Omega \pm \dot\lambda'$ is a constant, we effect the change of variables

$$\omega \pm \Omega \pm \lambda' = (\dot\omega \pm \dot\Omega \pm \dot\lambda')t + (\omega_0 \pm \Omega_0 + \lambda_0') \qquad (24a)$$

where $\dot\omega$, $\dot\Omega$, and $\dot\lambda'$ are the mean angular rates of variation of the argument of the perigee, of the node longitude, and of the sun's longitude, respectively, in the ecliptic, determined by the formulas:

$$\begin{aligned} \dot\omega &= \frac{1}{2}J\frac{n}{a^2(1-e^2)^2}(5\cos^2 i - 1) \\ \dot\Omega &= -J\frac{n}{a^2(1-e^2)^2}\cos i \\ \dot\lambda' &= +0.98565(\text{deg}/24\text{ hr}) \end{aligned} \qquad (25)$$

If not a single coefficient of t in the arguments of expansion (24) vanishes, then the first-order perturbations can be obtained by integration of differential equations (24), on whose right-hand sides the orbital elements have been replaced by their mean values, with allowance for the secular motion of the node and perigee due to the earth's oblateness. After integration, the right-hand sides will assume the form

$$\int A_{k\,\cos}^{\;\sin}(\omega\pm\Omega\pm\lambda')dt = \int A_{k\,\cos}^{\;\sin}[(\dot\omega\pm\dot\Omega\pm\dot\lambda')t + (\omega_0\pm\Omega_0\pm\lambda_0')]dt = \mp\frac{A_k}{\dot\omega\pm\dot\Omega\pm\dot\lambda'}{}_{\sin}^{\cos}(\omega\pm\Omega\pm\lambda') + C \qquad (26)$$

Using (26), we shall write the formula for the osculating elements:

$$e = e_0 + \frac{3}{2}\frac{Fna^2\sqrt{1-e^2}}{G\mathfrak{M}}\times\left[\frac{\cos^2\frac{i}{2}\sin^2\frac{\epsilon}{2}}{\dot\omega+\dot\Omega+\dot\lambda'}\cos(\omega+\Omega+\lambda') + \frac{\sin^2\frac{i}{2}\sin^2\frac{\epsilon}{2}}{\dot\omega-\dot\Omega-\dot\lambda'}\times\cos(\omega-\Omega-\lambda') + \frac{\cos^2\frac{i}{2}\cos^2\frac{\epsilon}{2}}{\dot\omega+\dot\Omega-\dot\lambda'}\cos(\omega+\Omega-\lambda') + \frac{\sin^2\frac{i}{2}\cos^2\frac{\epsilon}{2}}{\dot\omega-\dot\Omega+\dot\lambda'}\cos(\omega-\Omega+\dot\lambda') - \frac{1}{2}\frac{\sin i\sin\epsilon}{\dot\omega+\dot\lambda'}\cos(\omega+\lambda') + \frac{1}{2}\frac{\sin i\sin\epsilon}{\dot\omega+\dot\lambda'}\cos(\omega-\lambda')\right]$$

$$i = i_0 + \frac{3}{4}\frac{Fna^2e}{G\mathfrak{M}\sqrt{1-e^2}}\left[\frac{\sin i\sin^2\frac{\epsilon}{2}}{\dot\omega+\dot\Omega+\dot\lambda'}\cos(\omega+\Omega+\lambda') - \frac{\sin i\sin^2\frac{\epsilon}{2}}{\dot\omega-\dot\Omega-\dot\lambda'}\cos(\omega-\Omega-\lambda') + \frac{\sin i\cos^2\frac{\epsilon}{2}}{\dot\omega+\dot\Omega-\dot\lambda'}\times\cos(\omega+\Omega-\lambda') - \frac{\sin i\cos^2\frac{\epsilon}{2}}{\dot\omega-\dot\Omega+\dot\lambda'}\cos(\omega-\Omega+\lambda') + \frac{\cos i\sin\epsilon}{\dot\omega+\dot\lambda'}\cos(\omega+\lambda') - \frac{\cos i\sin\epsilon}{\dot\omega-\dot\lambda'}\cos(\omega-\lambda')\right] \qquad (27)$$

$$\Omega = \Omega_0 + \frac{3}{4}\frac{Fna^2e}{G\mathfrak{M}\sin i\sqrt{1-e^2}}\left[\frac{\sin i\sin^2\frac{\epsilon}{2}}{\dot\omega+\dot\Omega+\dot\lambda'}\times\sin(\omega+\Omega+\lambda') - \frac{\sin i\sin^2\frac{\epsilon}{2}}{\dot\omega-\dot\Omega-\dot\lambda'}\sin(\omega-\Omega-\lambda') + \frac{\sin i\cos^2\frac{\epsilon}{2}}{\dot\omega+\dot\Omega-\dot\lambda'}\sin(\omega+\Omega-\lambda') - \frac{\sin i\cos^2\frac{\epsilon}{2}}{\dot\omega-\dot\Omega+\dot\lambda'}\times\sin(\omega-\Omega+\lambda') + \frac{\cos i\sin\epsilon}{\dot\omega+\dot\lambda'}\sin(\omega+\lambda') - \frac{\cos i\sin\epsilon}{\dot\omega-\dot\lambda'}\sin(\omega-\lambda')\right]$$

$$\omega = \omega_0 - \frac{3}{2}\frac{Fna^2\sqrt{1-e^2}}{G\mathfrak{M}}\left[\frac{\cos^2\frac{i}{2}\sin^2\frac{\epsilon}{2}}{\dot\omega+\dot\Omega+\dot\lambda'}\times \sin(\omega+\Omega+\lambda') + \frac{\sin^2\frac{i}{2}\sin^2\frac{\epsilon}{2}}{\dot\omega-\dot\Omega-\dot\lambda'}\sin(\omega-\Omega-\lambda') + \frac{\cos^2\frac{i}{2}\sin^2\frac{\epsilon}{2}}{\dot\omega+\dot\Omega-\dot\lambda'}\sin(\omega+\Omega-\lambda') + \frac{\sin^2\frac{i}{2}\cos^2\frac{\epsilon}{2}}{\dot\omega-\dot\Omega+\dot\lambda'}\times \sin(\omega-\Omega+\lambda') - \frac{1}{2}\frac{\sin i\sin\epsilon}{\dot\omega+\dot\lambda'}\sin(\omega+\lambda') + \frac{1}{2}\frac{\sin i\sin\epsilon}{\dot\omega-\dot\lambda'}\sin(\omega-\lambda')\right] - \Omega$$

$$M_0+\omega = (M_0+\omega_0) + (1-\sqrt{1-e^2})\delta\omega - \cos i\sqrt{1-e^2}\,\delta\Omega - \frac{3Fa^2e}{G\mathfrak{M}}\left[\frac{\cos^2\frac{i}{2}\sin^2\frac{\epsilon}{2}}{\dot\omega+\dot\Omega+\dot\lambda'}\times \sin(\omega+\Omega+\lambda') + \frac{\sin^2\frac{i}{2}\sin^2\frac{\epsilon}{2}}{\dot\omega-\dot\Omega-\dot\lambda'}\sin(\omega-\Omega-\lambda') + \frac{\cos^2\frac{i}{2}\cos^2\frac{\epsilon}{2}}{\dot\omega+\dot\Omega-\dot\lambda'}\sin(\omega+\Omega-\lambda') + \frac{\sin^2\frac{i}{2}\cos^2\frac{\epsilon}{2}}{\dot\omega-\dot\Omega+\dot\lambda'}\times \sin(\omega-\Omega+\lambda') - \frac{1}{2}\frac{\sin i\sin\epsilon}{\dot\omega+\dot\lambda'}\sin(\omega+\lambda') + \frac{1}{2}\frac{\sin i\sin\epsilon}{\dot\omega-\dot\lambda'}\sin(\omega-\lambda')\right] \quad (27)$$

Let us now consider the case when the coefficients of t in the arguments of expansion (24) can become small, and the perturbations of the elements can become large. This is called the resonance case. The amplitudes, frequencies, and periods of the perturbations are determined by the orbital elements.

Using the foregoing formulas, let us now consider the main effects in the variation of elements in the resonance case, i.e., when the conditions

$$\begin{aligned} \dot\omega+\dot\Omega+\dot\lambda' &= 0 \quad \text{(I)}\\ \dot\omega-\dot\Omega-\dot\lambda' &= 0 \quad \text{(II)}\\ \dot\omega+\dot\Omega-\dot\lambda' &= 0 \quad \text{(III)}\\ \dot\omega-\dot\Omega+\dot\lambda &= 0 \quad \text{(IV)}\\ \dot\omega+\dot\lambda' &= 0 \quad \text{(V)}\\ \dot\omega-\dot\lambda' &= 0 \quad \text{(VI)} \end{aligned} \qquad (28)$$

are satisfied. It is of interest to calculate the values of the orbital inclination i, corresponding to the resonance conditions for one of the arguments, by using formulas (25). Such values of the inclination we shall call critical. In order to determine the critical i for the first argument, it is necessary to solve the quadratic equation

$$5\cos^2 i - 2\cos i - \left[1 - \frac{2\dot\lambda' a^2(1-e^2)^2}{In}\right] = 0 \qquad (29)$$

whose roots are determined by the formula

$$\cos i_{1,2} = \frac{2 \pm \sqrt{4 + 20(1-a)}}{10} \qquad (30)$$

where $\alpha = [2\dot\lambda' a^2(1-e^2)^2]/Jn$ is a parameter whose values can be tabulated for a number of values of e, for 500 km $\leqslant h_{\text{mean}} \leqslant$ 10000 km. We shall determine, by formula (30), two regions of critical-inclination values for various altitudes and eccentricities. Thus, e.g., to the first argument, the positive root corresponds to the region $46°.6 \leqslant i_1 \leqslant 78°.5$, i.e., the root does not leave the region of direct satellite motion. The region $78°.5 \leqslant i_2 \leqslant 107°.8$ corresponds to the negative root, i.e., resonance can occur also for satellites with retrograde motion. For both roots we have asymptotic approximation to the values $i_1 = 46°.4$ (from above) and $i_2 = 107°.8$ (from below), which corresponds to $\alpha = 0$ in formula (30). For greater clarity, let us plot the variation of the inclinations i as

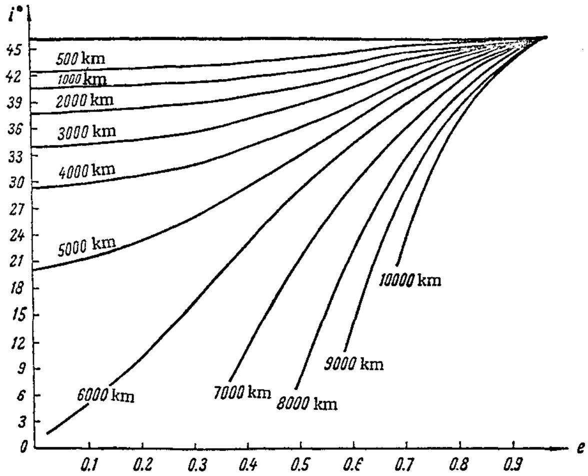


Fig. 3 Critical inclinations for condition $\dot\omega - \dot\Omega - \dot\lambda' = 0$.

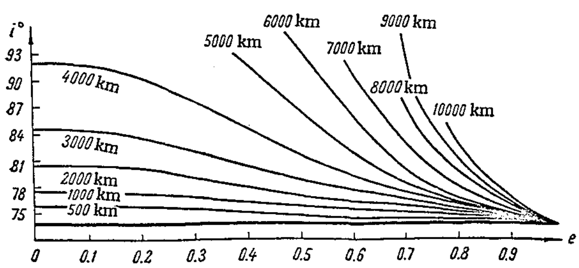


Fig. 2 Critical inclinations for condition $\dot\omega + \dot\Omega + \dot\lambda' = 0$.

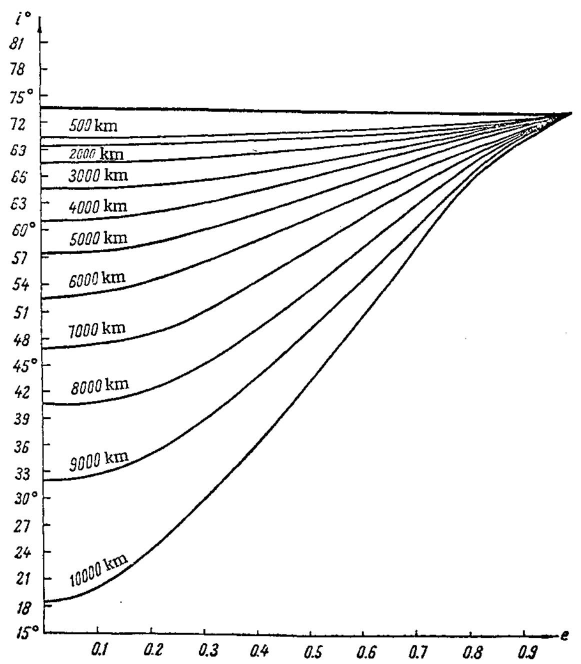


Fig. 4 Critical inclinations for condition $\dot\omega + \dot\Omega - \dot\lambda' = 0$.

Table 1 Critical inclinations under resonance conditions

$h_{mean}(km)$	$\dot\omega + \dot\Omega + \dot\lambda'$	$\dot\omega - \dot\Omega - \dot\lambda$	$\dot\omega + \dot\Omega - \dot\lambda'$	$\dot\omega - \dot\Omega + \dot\lambda'$	$\dot\omega + \dot\lambda'$	$\dot\omega - \dot\lambda'$
500	49°	70°	42°	76°	59°	67°
1000	52	69	40	78	57	67
2000	55	67	38	81	56	71

Table 2 Arguments of long-period perturbations

	Vanguard 1 1958 β_2 ($i = 34°$)	Third (Soviet) AES 1958 δ_2 ($i = 65°$)	Explorer 7 1959 ι_1 ($i = 50°$)	Transit 2A 1960 η_1 ($i = 67°$)	Echo 1 1960 ι_1 ($i = 47°$)	Tiros 2 1960 π_1 ($i = 48°$)	Explorer 9 1961 δ_1 ($i = 39°$)	Discoverer 20 1961 ϵ_1 ($i = 81°$)
$\dot\omega + \dot\Omega + \dot\lambda'$	+2.374	−1.896	+0.207	−2.293	+0.867	+0.529	+2.097	−3.492
$\dot\omega - \dot\Omega - \dot\lambda'$	+6.422	+1.180	+6.579	+0.853	+5.079	+7.845	+7.413	−3.088
$\dot\omega + \dot\Omega - \dot\lambda'$	+0.402	−3.868	−1.765	−4.265	−1.105	−1.443	+0.125	−5.464
$\dot\omega - \dot\Omega + \dot\lambda'$	+8.394	+3.152	+8.551	+2.825	+7.051	+9.817	+9.385	−1.116
$\dot\omega + \dot\lambda'$	+5.384	+0.628	+4.379	+0.266	+3.959	+5.173	+5.741	−2.304
$\dot\omega - \dot\lambda'$	+3.412	−1.344	+2.407	−1.706	+1.987	+3.201	+3.769	−4.276

a function of eccentricity variation for various values of mean satellite-altitude over the earth's surface from 500 to 10,000 km (Fig. 2). The graph represents the region of direct satellite motion only.

For the other arguments, it is necessary to solve other (analogous) quadratic equations to determine the region of critical inclinations (Figs. 3–6). With the help of these graphs and knowing the eccentricity, mean altitude, and orbital inclination of the satellite, it is possible to determine whether the satellite is in the resonance region. Taking into account that the majority of existing satellites have an altitude not exceeding 2000 km, let us tabulate critical inclinations i for each of the 6 arguments of (28) (Table 1).

The resonance conditions have a simple physical interpretation. We shall assume that the elements are referred to the ecliptic; then $\lambda' - \Omega$ is the sun's longitude with respect to the line of nodes and ω is the perigee longitude with respect to the same line. Condition (III) or the equality $\dot\omega = \dot\lambda' - \dot\Omega$ signifies, for example, that the mean angular velocities of the sun and of the perigee are equal and that the perigee follows the sun all the time, remaining close to it. In case (IV), $\dot\omega = -(\dot\lambda - \dot\Omega)$, i.e., the sun and the perigee move with the same angular velocities, but in different directions. The critical inclination, corresponding to the vanishing of the expression $\dot\omega + \dot\Omega - \dot\lambda'$, is 40° for a satellite altitude of about 1000 km.

Table 2 lists the values of the coefficients of the secular terms of the arguments of (28), calculated by formulas (25) for eight existing or past satellites with different orbital elements. If the orbital inclination of the satellite is close to some value which is critical for the given altitude (this can be approximately determined from Table 1), then it is evident that the corresponding coefficient will be the smallest. These coefficients are underlined in Table 2. Indeed, e.g., for the satellites "Vanguard 1" ($i = 34°$) and "Explorer 9" ($i = 39°$) the smallest coefficient was found to be the coefficient $\dot\omega + \dot\Omega - \dot\lambda'$, and for the satellites "Explorer 7" ($i = 50°$), "Echo 1" ($i = 47°$), "Tiros 2" ($i = 48°$), the coefficient $\dot\omega + \dot\Omega + \dot\lambda'$, etc.

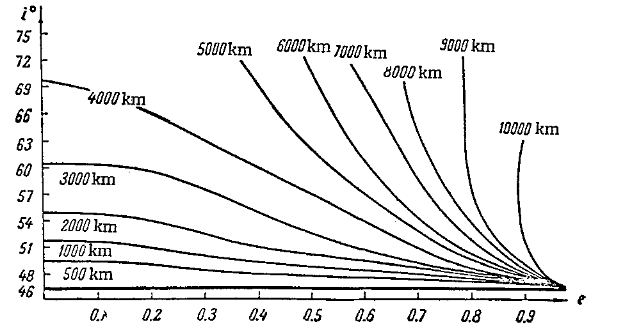


Fig. 5 Critical inclinations for condition $\dot\omega - \dot\Omega + \dot\lambda' = 0$.

Let us now consider the perturbation amplitudes, i.e., coefficients of type $[\cos^2(i/2)\sin^2(\epsilon/2)]/(\dot\omega + \dot\Omega + \dot\lambda')$, without multiplying them by the quantity $\frac{3}{2}(Fna^2\sqrt{1 - e^2})/G\mathfrak{M}$ or $\frac{3}{4}Fna^2e/(G\mathfrak{M}\sqrt{1 - e^2})$, which is constant for each satellite. It is evident that for satellites with an inclination close to 40°, the long-period term with argument $\omega + \Omega - \lambda'$ has an amplitude considerably greater than the amplitudes of the other terms (Tables 3 and 4). The terms with this argument are the predominant ones in the expansions for the perturbations due to radiation pressure; this applies to the satellites "Vanguard 1" (1958 β_2) and "Echo 1" (1960 ι_1). By multi-

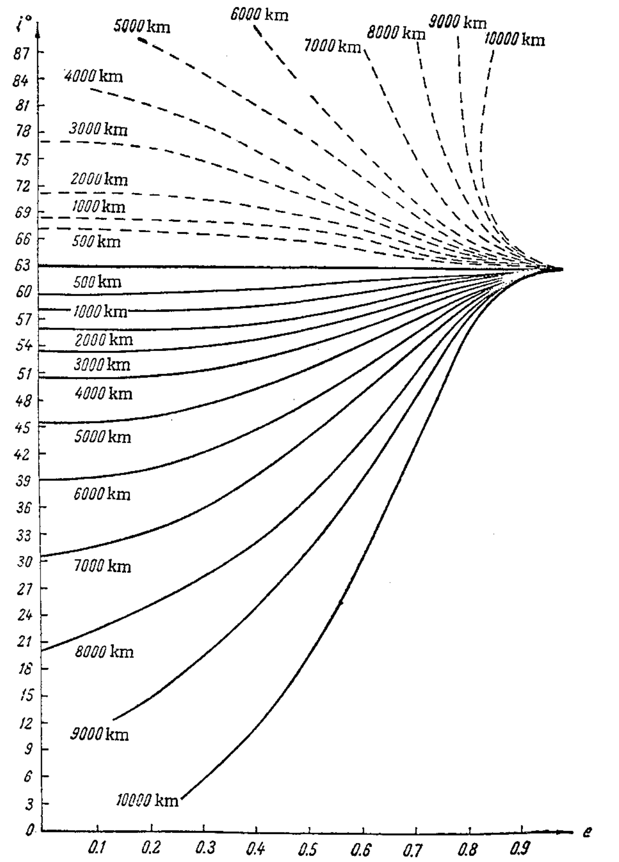


Fig. 6 Critical inclinations for condition $\dot\omega - \dot\lambda' = 0$ (dashed curves) and for condition $\dot\omega + \dot\lambda' = 0$ (solid curves).

Table 3 Coefficients of long-period perturbations in e and π

	Vanguard 1, 1958, β_2	Third (Soviet) AES, 1958, δ_2	Explorer 7, 1959, ι_1	Transit 2A, 1960, η_1	Echo 1, 1960, ι_1	Tiros 2, 1960, π_1	Explorer 9, 1961, δ_1	Discoverer 20 1961, ϵ_1
$\frac{\cos^2(i/2)\ \sin^2(\epsilon/2)}{\dot\omega + \dot\Omega + \dot\lambda'}$	+0.0159	−0.0154	+0.1630	−0.0125	+0.0398	+0.0648	+0.0175	−0.0068
$\frac{\sin^2(i/2)\ \sin^2(\epsilon/2)}{\dot\omega = \dot\Omega = \dot\lambda'}$	+0.0006	+0.0101	+0.0011	+0.0146	+0.0013	+0.0009	+0.0006	−0.0056
$\frac{\cos^2(i/2)\ \cos^2(\epsilon/2)}{\dot\omega + \dot\Omega - \dot\lambda'}$	+2.1766	−0.1758	−0.4446	−0.1566	−0.7279	−0.5520	+6.8152	−0.1015
$\frac{\sin^2(i/2)\ \cos^2(\epsilon/2)}{\dot\omega - \dot\Omega + \dot\lambda'}$	+0.0099	+0.0882	+0.0202	+0.1026	+0.0218	+0.0165	+0.0113	−0.3613
$\frac{(1/2)\sin i\ \sin\epsilon}{\dot\omega + \dot\lambda'}$	+0.0208	+0.2876	+0.0350	+0.6872	+0.0369	+0.0288	+0.0217	−0.0853
$\frac{(1/2)\sin i\ \sin\epsilon}{\dot\omega - \dot\lambda'}$	+0.0328	−0.1344	+0.0636	−0.1072	+0.0735	+0.0465	+0.0331	−0.0460

Table 4 Coefficients of long-period perturbations in i and Ω

	Vanguard 1, 1958, β_2	Third (Soviet) AES, 1958, δ_2	Explorer 7, 1959, ι_1	Transit 2A, 1960, η_1	Echo 1, 1960, ι_1	Tiros 2, 1960, π_1	Explorer 9, 1961, δ_1	Discoverer 20 1961, ϵ_1
$\frac{\sin i\ \sin^2(\epsilon/2)}{\dot\omega + \dot\Omega + \dot\lambda'}$	+0.0098	−0.0197	+01.531	−0.0165	+0.0348	+0.0584	+0.0123	−0.0117
$\frac{\sin i\ \sin^2(\epsilon/2)}{\dot\omega - \dot\Omega - \dot\lambda'}$	+0.0036	+0.0317	+0.0048	+0.0444	+0.0060	+0.0039	+0.0035	−0.0132
$\frac{\sin i\ \cos^2(\epsilon/2)}{\dot\omega + \dot\Omega - \dot\lambda'}$	+1.3413	−0.2248	−0.4178	−0.2064	−0.6364	−0.4972	+4.8080	−0.1731
$\frac{\sin i\ \cos^2(\epsilon/2)}{\dot\omega - \dot\Omega + \dot\lambda'}$	+0.0642	+0.2759	+0.0862	+0.3116	+0.0997	+0.0731	+0.0640	−0.8477
$\frac{\cos i\ \sin\epsilon}{\dot\omega + \dot\lambda'}$	+0.0611	+0.2661	+0.0580	+0.5902	+0.0683	+0.0510	+0.0540	−0.0273
$\frac{\cos i\ \sin\epsilon}{\dot\omega - \dot\lambda'}$	+0.0964	−0.1243	+0.1056	−0.0920	+0.1360	+0.0824	+0.0822	−0.0147

plying the amplitudes by the coefficients $\frac{3}{4}Fe/(ha\sqrt{1-e^2})$ and $\frac{3}{2}(F\sqrt{1-e^2})/na$, which are constant for each satellite [F being calculated by formula (6)], we obtain the perturbations due to radiation pressure. Thus, e.g., the eccentricity perturbations of the satellite "Vanguard 1" have the form

$$\begin{aligned}\delta e = {} & 0.0208\cdot 10^{-4}\cos(\omega + \Omega + \lambda') + \\ & 0.0008\cdot 10^{-4}\cos(\omega - \Omega - \lambda') + \\ & 2.8460\cdot 10^{-4}\cos(\omega + \Omega - \lambda') + \\ & 0.0129\cdot 10^{-4}\cos(\omega - \Omega + \lambda') - \\ & 0.0272\cdot 10^{-4}\cos(\omega + \lambda') + \\ & 0.0429\cdot 10^{-4}\cos(\omega - \lambda') \end{aligned} \quad (30a)$$

The period of the term with argument $\omega + \Omega - \lambda'$ is about 890 days, and this perturbation causes a perigee-height variation of about 2.4 km during that period. The perturbations of i and Ω have the same period as e, since the expansions were carried out in the same arguments.

The perturbations in the elements of the satellite "Vanguard 1" yield satisfactory agreement with theory if they are considered in conjunction with the lunar and solar perturbations.[13, 19] In Fig. 7 (Musen[13]), curve A corresponds to the calculated lunar-solar effect on the perigee-height variation with an amplitude of 2 km and a period of 450 days. Curve B corresponds to the observed values of perigee height, i.e., it includes the radiation-pressure effect.

Analogous expansions for the satellite "Echo 1" yield, in e, the main term $7.0539\cdot 10^{-2}\cos(\omega + \Omega - \lambda')$ with a period of 326 days, which corresponds to an eccentricity variation of about $2\cdot 10^{-4}$ per day for the first few days of satellite life (lateron A/m changes). The perigee-height variation is, in this case, about 2 km per day. The calculated values are in agreement with the experimental data, processed by Shaprio and Jones.[16]

3. Earth's Shadow Effect

In studying the effect of radiation pressure on satellite motion, it is necessary to take into consideration that, in general, the solar light is screened by the earth, and that the radiation-pressure is a discontinuous function, unlike the gravitational force which acts continuously. If the satellite enters the earth's shadow, from time to time, then, in addition to the long-period terms in the expansions for its orbital

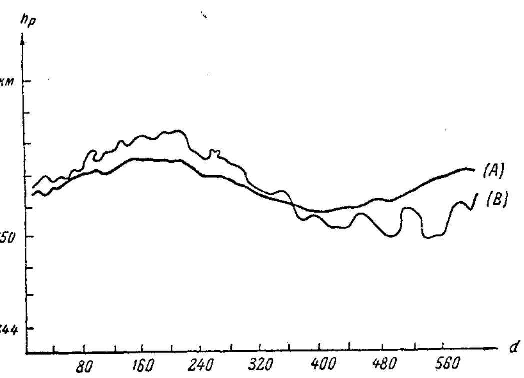


Fig. 7 Comparison of lunar-solar effect with effect of radiation pressure on variation of perigee height h_p of satellite Vanguard 1 (d refers to day of launching).

elements, the short-period terms also assume considerable importance. If the satellite enters the shadow irregularly, the derivation of analytic formulas for the perturbations becomes particularly difficult, since in this case no energy integral exists, and secular terms appear in the semimajor axis.

If the shadow boundary is determined, on the satellite orbit, by the points with eccentric anomalies E_1 and E_2 (the satellite enters the shadow at the point E_1), then the formulas for first-order perturbations over one orbital period have the following explicit form[8]:

$$\delta a = \frac{2a^3F}{G\mathfrak{M}}\left(S\cos E + T\sqrt{1-e^2}\sin E\right)\Big|_{E_2}^{E_1}$$

$$\delta e = \frac{a^2F\sqrt{1-e^2}}{G\mathfrak{M}}\left[\frac{1}{4}S\sqrt{1-e^2}\cos 2E\Big|_{E_2}^{E_1} + \left(T - 2e\cos E + \frac{1}{4}\sin 2E\right)\Big|_{E_2}^{E_1} + \frac{3}{2}\int_{E_2}^{E_1} T\,dE\right]$$

$$\delta i = \frac{a^2FW}{G\mathfrak{M}\sqrt{1-e^2}}\left\{\left[(1+e^2)\sin E - \frac{e}{4}\sin 2E\right]\Big|_{E_2}^{E_1}\times \cos\omega + \sqrt{1-e^2}\left(\cos E - \frac{e}{4}\cos 2E\right)\sin\omega\Big|_{E_2}^{E_1} - \frac{3}{2}e\int_{E_2}^{E_1} W\cos\omega\,dE\right\} \quad (31)$$

$$\sin i\,\delta\Omega = \frac{a^2FW}{G\mathfrak{M}\sqrt{1-e^2}}\left\{\left[(1+e^2)\sin E - \frac{e}{4}\sin 2E\right]\Big|_{E_2}^{E_1}\times \sin\omega - \sqrt{1-e^2}\left(\cos E - \frac{e}{4}\cos 2E\right)\cos\omega\Big|_{E_2}^{E_1} - \frac{3}{2}e\int_{E_2}^{E_1} W\sin\omega\,dE\right\}$$

$$\delta\omega = -\cos i\,\delta\Omega + \frac{a^2F\sqrt{1-e^2}}{G\mathfrak{M}e}\times\left[S\left(e\sin E + \frac{1}{4}\sin 2E\right)\Big|_{E_2}^{E_1} + T\sqrt{1-e^2}\times\left(e\cos E - \frac{1}{4}\cos 2E\right)\Big|_{E_2}^{E_1} - \frac{3}{2}\int_{E_2}^{E_1} S\,dE\right]$$

$$\delta M = -\frac{3}{2}\int_0^{2\pi}\frac{\delta a}{a}\,dM - \sqrt{1-e^2}\,\delta\omega - \sqrt{1-e^2}\cos i\,\delta\Omega - \frac{2a^2F}{G\mathfrak{M}}\left\{\left[(1+e^2)\sin E - \frac{e}{4}\sin 2E\right]\Big|_{E_2}^{E_1} - \frac{T\sqrt{1-e^2}}{G\mathfrak{M}}\left(\cos E - \frac{e}{4}\cos 2E\right)\Big|_{E_2}^{E_1} - \frac{3}{2}e\int_{E_2}^{E_1} S\,dE\right\}$$

These formulas are obtained after integration, with respect to E, of differential equations (21), where the quantities $S(v)$ and $T(v)$ on their right-hand sides are expressed in the form

$$\begin{aligned} S(v) &= S(0)\cos v + T(0)\sin v \\ T(v) &= T(0)\cos v - S(0)\sin v \end{aligned} \quad (32)$$

In formulas (31), S and T are the values of $S(v)$ and $T(v)$ for $v = 0$.

If the satellite does not enter the shadow during one revolution, i.e., if the orbit is evenly illuminated, then we arrive at the case already considered in Sec. 2: All the terms which depend explicitly on E vanish—in particular, δa vanishes completely, and for the other elements we obtain, by passing again to the variable t, the equations of Sec. 2.

If the satellite enters the shadow during one orbit, then the problem reduces to the determination of the limits of integration E_1 and E_2 or, by effecting the substitution

$$\tan\frac{v}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$$

to the determination of v_1 and v_2, i.e., the solution of the shadow equation, which is derived next. Since the analytical solution of this equation involves expansion in series in powers of the eccentricity, it is only possible for circular orbits; for elliptical orbits the solution is difficult, as a result of the weak convergence. In the latter case, it is more convenient to solve the shadow equation numerically for each orbit separately, as suggested by Kozai.[8]

Let us now consider the shadow equation and an analytical method for its solution for some particular types of orbital shape and orientation, as suggested by Wyatt.[20] In this connection, it is of interest to examine the shadow effect in the case of the radiation-pressure influence on the secular acceleration of the satellite, i.e., on the quantity $\Delta P/P$ (the variation of the satellite period during one orbit, referred to the period).

We shall assume, as before, that the radial perturbing acceleration under the effect of the radiation pressure is determined by the formula

$$F = \mathfrak{P}r\,\frac{A}{m} \quad (33)$$

For simplicity, let us assume that during one revolution $\bar{F}$ is constant with respect to the satellite orbit, and that it is possible to neglect such effects as the variation of the solar constant, the variation of the distance to the sun, the motion of the sun with respect to α and δ, the relativistic Poynting-Robertson effect, and the effect of reradiation from the earth. For the semimajor axis variation, we have the formula

$$\frac{da}{dt} = \frac{Pe\sin v}{\pi\sqrt{1-e^2}}FS + \frac{P(1+e\cos v)}{\pi\sqrt{1-e^2}}FT \quad (34)$$

where P is the orbital period, e the eccentricity, v the true anomaly, and FS and FT are, respectively, the radial and transverse components of the perturbing acceleration (see Sec. 2). Transforming (34), by the use of the law of areas and the orbital equation in polar coordinates, we obtain

$$\frac{dP}{dv} = \frac{dP}{da}\frac{da}{dt}\frac{dt}{dv} = \frac{3Pa^2(1-e^2)}{G\mathfrak{M}}\left[\frac{FSe\sin v + FT(1+e\cos v)}{(1+e\cos v)^2}\right] \quad (35)$$

Let us again write the values of the perturbing-acceleration components, but in a somewhat different form than in Sec. 2. In Fig. 8 the xy plane coincides with the orbital plane, and the x axis coincides with the line of intersection of the orbital plane and the plane perpendicular to it which contains the sun. The origin E coincides with the earth's center, P is the instantaneous position of the satellite, Q the orbital perigee, v the true anomaly, and β the angular distance of the perigee

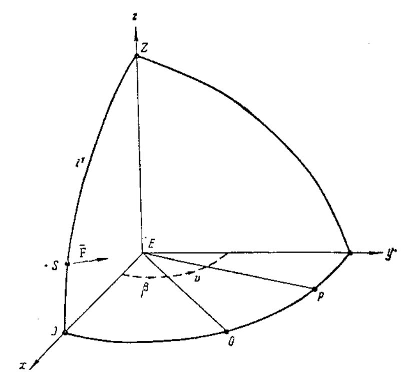


Fig. 8

from the x axis. It is evident that the perturbing acceleration, i.e., the component of $\bar{F}$ which lies in the orbital plane, is equal to $F \sin i'$, where i' is the angular distance of the sun from the orbital pole. Expressing the obtained quantity in terms of the radical and transverse components, we obtain

$$\begin{aligned} \bar{F}S &= -\bar{F}\sin i' \cos(v + \beta) \\ \bar{F}T &= +\bar{F}\sin i' \sin(v + \beta) \end{aligned} \tag{36}$$

In formulas (36), it is assumed that $\bar{F}$, i', and β are constant during one revolution. Substituting (36) in (35) and integrating over the entire orbit, we obtain the value of the secular acceleration due to the radiation pressure

$$\frac{\Delta P}{P} = \frac{1}{P}\int_0^{2\pi} \frac{dP}{dv}\,dv = \frac{3a^2(1 - e^2)F \sin i'}{G\mathfrak{M}} \times \left[\int_0^{v_1} f(v,\beta)dv + \int_v^{2\pi} f(v,\beta)dv\right] \tag{37}$$

where

$$f(v,\beta) = \frac{\cos\beta \sin v + \sin\beta(1 + \cos v)}{(1 + e\cos v)^2}$$

and v_1 and v_2 are the values of the true anomaly which determine the shadow boundary. After integration we obtain

$$\left.\frac{e^{-1}\cos\beta + \sin\beta \sin v}{1 + e\cos v}\right|_0^{v_1} - \left.\frac{e^{-1}\cos\beta + \sin\beta\sin v}{1 + e\cos v}\right|_{v_1}^{2\pi} = \left.\frac{\cos(\beta + v)}{1 + e\cos v}\right|_{v_1}^{v_2} \tag{38}$$

Substituting (38) in (37) and denoting the ratio of perigee distance to earth radius q/R_0 by K ($K > 1$), we obtain

$$\frac{\Delta P}{P} = \frac{F}{G\mathfrak{M}} K^2 \frac{1 + e}{1 - e} \sin i' \left[\frac{\cos(\beta + v)}{1 + e\cos v}\right]\Bigg|_{v_1}^{v_2} \tag{39}$$

If the orbit is uniformly illuminated, then one has to set $v_1 = v_2$ in formula (38). In this case, the zero result shows that the radiation pressure does not affect the orbital period. Thereby the action of the radiation-pressure force is analogous to that of the gravitational force.

If the satellite enters the earth's shadow, then it is necessary to determine the limits of integration v_1 and v_2. Wyatt considers the earth's shadow as a right circular cylinder of radius R_0, whose intersection with the orbital plane of the satellite is a semiellipse (Fig. 9), described by the equation

$$x^2\cos^2 i' + y^2 = R_0^2 \qquad (x \leqslant 0) \tag{40}$$

Passing to polar coordinates and using the orbit equation, we shall find that v_1 and v_2 are two roots of the equation

$$\frac{R_0^2}{1 - \sin^2 i' \cos^2(\beta + v)} = \frac{a^2(1 - e^2)^2}{(1 + e\cos v)^2}\left(\frac{\pi}{2} \leqslant \beta + v \leqslant \frac{3\pi}{2}\right) \tag{41}$$

or

$$(1 + e\cos v)^2 = K^2(1 + e^2)[1 - \sin^2 i' \cos^2(\beta + v)]$$

Formula (41) is called the shadow equation; it has no roots if the entire orbit is illuminated, a single root $v_1 = v_2$ if the orbit touches the shadow at a single point, and two roots in

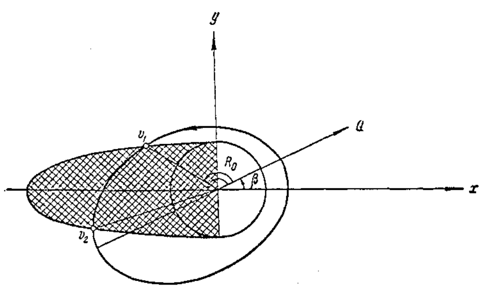


Fig. 9

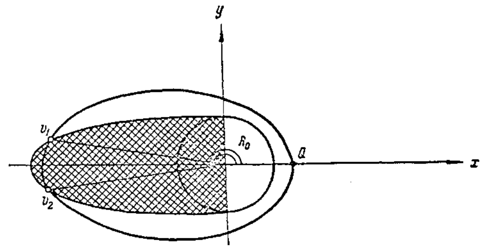


Fig. 10

the general case of satellite entry into the shadow. The limits of integration can be obtained by graphical solution of Eq. (41), when i' and β are calculated for several days of observation. However, since, in the general case, the limits of integration (which are the general solution of the shadow equation) depend on several arbitrary parameters, the problem of integration becomes greatly complicated. Therefore, let us consider some particular types of orbital shape and orientation.

1) If $i' = 0$, i.e., an orbital plane of arbitrary eccentricity is normal to the line earth-sun, then the satellite will never be in the shadow. If the satellite does not enter the shadow for $i' \neq 0$, i.e., the sun is steadily visible above the horizon as seen from the satellite, then the integral is evaluated from 0 to 2π and $\Delta P/P = 0$.

2) In the case of a circular orbit of any orientation, the radiation pressure also has no effect on the secular acceleration, since by virtue of symmetry $\beta + v_2 = 2\pi - (\beta + v_1)$ and from (39) it follows that $\Delta P/P = 0$, although the satellite enters the shadow. The momenta imparted to the satellite on moving toward and away from the sun balance each other.

3) The zero effect applies also to orbits of arbitrary e, if the perigee lies on the x axis, i.e., if $\beta = 0, \pi$ (Fig. 10). By axial symmetry, we have $v_2 = 2\pi = v_1$, which again yields $\Delta P/_0 = 0$.

4) Let us now consider the case of maximum asymmetry, when $i' = \pi/2$, $\beta = \pi/2, 3\pi/2$. Then the solutions of the shadow equation are

$$\cos v_1 = [K(1 + e) - e]^{-1}, \quad \cos v_2 = -[K(1 - e) + e]^{-1}$$

$$\frac{\Delta P}{P} = \pm\frac{F}{G\mathfrak{M}}\frac{K\sqrt{1 + e}}{1 - e} \times \left[\sqrt{(K^2 - 1) + e(K^2 + 1) + 2Ke} - \sqrt{(K^2 - 1) + e(K^2 + 1) - 2Ke}\right] \tag{42}$$

Formula (42) gives the maximum value of $\Delta P/P$ that could be obtained from radiation pressure, since for $i' = \pi/2$ the radiation pressure vector lies entirely in the orbital plane and the values $\beta = \pi/2, 3\pi/2$ yield the maximum effect of orbital asymmetry. For $\beta = \pi/2$ the period decreases with time and $\Delta P/P < 0$; for $\beta = 3\pi/2$ the period increases, $\Delta P/P > 0$, the acceleration being equal and of opposite sign.

5) Let us now consider the general solution of the shadow equation, in the form of power series in e, for the general case of an orbit with small e. We shall introduce the polar coordinates r and $\varphi = \beta + v$. From Fig. 11, where a circle of radius q has been drawn about the earth's center, we have

$$\angle JCA = \varphi_1 \qquad \angle JCB = \varphi_2 \qquad \angle JCA_0 = \varphi_{01} \qquad \angle JCB_0 = \varphi_{02}$$

Then $\varphi_1 = \varphi_{01} + u_1$, $\varphi_2 = \varphi_{02} + u_2$, where u_1 and u_2 are small for small e, whereas for $e = 0$, $u_1 = u_2 = 0$, and the shadow equation yields

$$\cos\varphi_{01} = \cos\varphi_{02} = -\frac{\sqrt{K^2 - 1}}{K\sin i'} \equiv -\nu$$

$$(\mu, \nu \geqslant 0) \tag{43}$$

$$\sin\varphi_{01} = -\sin\varphi_{02} = \frac{\sqrt{1 - K^2\cos^2 i'}}{K\sin i'} \equiv \mu$$

Expanding u_1 and u_2 in power series in e:

$$\begin{aligned} u_1 &= \mathfrak{U}_1 e + \mathfrak{U}_2 e^2 + \ldots + \mathfrak{U}_n e^n \\ u_2 &= \mathfrak{B}_1 e + \mathfrak{B}_2 e^2 + \ldots + \mathfrak{B}_n e^n \end{aligned} \qquad (44)$$

and substituting (44) in (41), let us compare the coefficients of equal powers in e; as a result one obtains $\mathfrak{U}_u$ and $\mathfrak{B}_n$. Expanding (39) in power series in e and introducing, in the obtained formulas, the values of $\mathfrak{U}_n$ and $\mathfrak{B}_n$, we shall obtain the secular acceleration due to the radiation pressure in the form of power series in e with known coefficients. Since the calculation of the coefficients of powers higher than the second is very cumbersome, this method is only applicable for small e. Finally, the general solution to second order in e inclusive, is obtained in the form

$$\frac{\Delta P}{P} = -\frac{F}{G\mathfrak{M}} \frac{2K^3 e\mu \sin\beta}{\sqrt{K^2 - 1}} \times \left[1 + 2e(1 + \nu\cos\beta) - \frac{e(1 + \nu^3\cos\beta)}{\mu^2(K^2 - 1)}\right] + 0(e^3) \qquad (45)$$

Formula (45) gives, for $e = 0$, the second case considered in the foregoing, and for $\beta = 0$, π the third case. For $i' = \pi/2$ and $\beta = \pi/2$, $3\pi/2$, we again obtain the asymmetric fourth case. Indeed, by expanding (42) in powers in e, and substituting the values of i' and β in (45), we obtain the same result

$$\frac{\Delta P}{P} = -\frac{F}{G\mathfrak{M}} \left\{\frac{2K^2 e}{\sqrt{K^2 - 1}}\left[1 + \frac{e(K^2 - 2)}{K^2 - 1}\right]\right\}$$

Wyatt considers, in more detail, the general solution for the case $i' = \pi/2$, i.e., when the orbital plane is parallel to the direction of the solar rays. This particular case is close to the fourth case, but since β is not determined, the asymmetry effect changes the magnitude of the secular acceleration, calculated by the formula

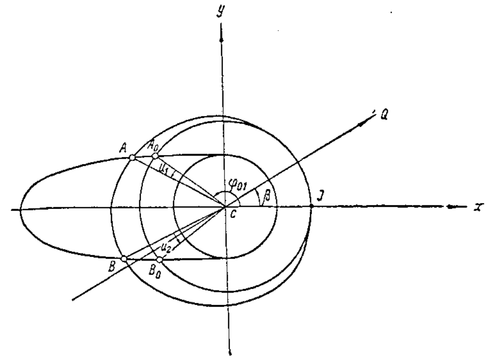
$$\frac{\Delta P}{P} = -\frac{F}{G\mathfrak{M}} eV(K,\beta) \times \left\{1 + e\left[\frac{K^2 - 2}{K^2 - 1} + \frac{2\sqrt{K^2 - 1}}{K}\cos\beta\right] + \ldots\right\} \qquad (46)$$

where $V(K,\beta) = 2K^2 \sin\beta/\sqrt{K^2 - 1}$. The variation of $V(K,\beta)$ is represented in Fig. 12, which permits one to estimate the first term of (46); the magnitude of the second term, as estimated by Wyatt, does not exceed 20% of the first. From the graph it is evident that, for any fixed perigee distance, the secular acceleration increases when β increases from 0 to $\pi/2$, where $\Delta P/P = 0$ for $\beta = 0$, and $\Delta P/P$ has a maximum for maximum asymmetry, i.e., for $\beta = \pi/2$, for any perigee distance. For any fixed β the secular acceleration has a relative maximum for $K \to 1$; then it drops to a relative minimum for an intermediate $K \approx 1.45$, and finally it increases again. Fig. 12 illustrates the effect of perigee asymmetry with respect to the line earth-sun on the secular acceleration of the satellite.

4. Effect of Radiation Pressure on Motion of Satellite "Echo 1"

As an example of a satellite whose motion is considerably affected by radiation pressure, let us consider the American satellite "Echo 1" (1960 ι_1). This satellite was launched on August 12, 1960 into an almost circular orbit ($e_0 = 0.010$) with an inclination of 47°.2 and a perigee and apogee height of 1500 and 1700 km, respectively; it was intended for experimental intra- and intercontinental radiocommunications. The launching of the satellite "Echo 1" offered, in fact, the first real opportunity for experimentally investigating the effect of radiation pressure on satellite motion in a circular orbit, since for "Echo 1" the area-to-mass ratio is about 1000 times larger then for all the other satellites. The satellite

Fig. 11



"Echo 1" is a gas-filled sphere 30.5 m in diameter, made of synthetic aluminized material "Mylar," 0.5 mm thick. The initial weight of the satellite, including the chemical which liberates the gas, was 70.4 kg; of this, the chemical weighed about 15 kg. The initial value of the area-to-mass ratio is $A_0/m_0 = 102$ cm²/g, but it increases subsequently as a result of loss of mass by gradual escape of gas through point holes in the balloon made during launching or punctured by micrometeorites. This loss of mass, which leaves the shape of the sphere unchanged while reducing its size, is difficult to predict; it can only be estimated approximately. The following estimates of mass loss of "Echo 1" fitted observation best: about 300 g per day during the first fortnight after launching, later on about 70 g per day.

The perturbations of the elements of the satellite "Echo 1" were determined by Parkinson, et al.[15] and Kozai[8] for direct solar radiation with allowance for the shadow effect. The following were considered to be perturbing forces: the harmonics in the expansion of the potential of the earth's gravitational field (from second to fifth), atmospheric drag, lunar-solar effects and, finally, the effect of reradiation from the ground. The latter effect is weak, since it turns out to be roughly 100 times smaller than the direct solar radiation-pressure effect (even if the hypothesis of specular reflection from the earth is accepted). Among the harmonics in the expansion of the potential only the second gives a substantial contribution, whereas the perturbations due to the other harmonics are weak; thus, e.g., the perturbation in the eccentricity due to the third harmonic is $-3 \cdot 10^{-5}$ per day, i.e., roughly 10 times less than the corresponding perturbation from radiation pressure. Only neutral atmospheric drag is considered, since the charge drag effect does not exceed the accuracy of observation even for a maximum balloon potential of about 100 v. Possible corrections due to the irregularity of the solar radiation flux, to the corpuscular radiation of the sun, to the appearance of a dynamic moment on collision with micrometerorites, to the satellite's rotation, and to the change

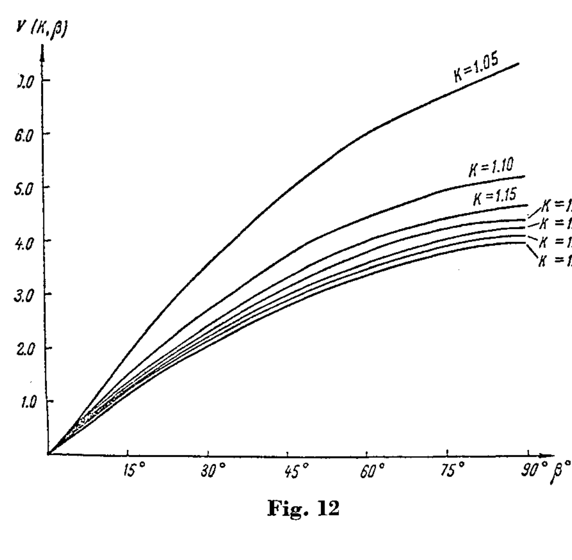


Fig. 12

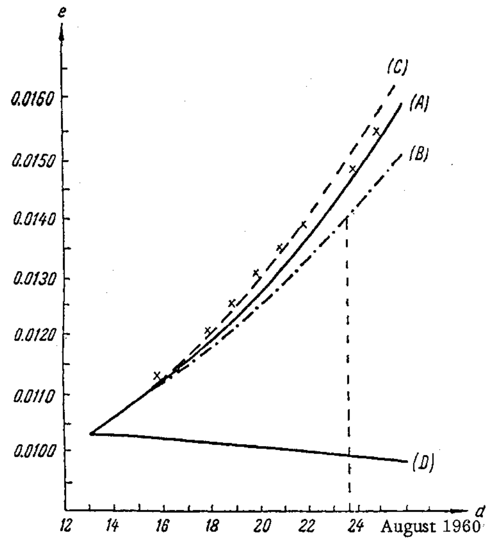


Fig. 13 Eccentricity variation of satellite Echo 1 over period August 12–25, 1960 (dashed line denotes time of first entry into earth's shadow).

in thermodynamic conditions on its surface, are also disregarded.

The perturbations of the satellite "Echo 1" due to radiation pressure were also determined by Shaprio and Jones,[16] who ascertained the perturbations for the first 12 days of life of "Echo 1" by processing accurate photographic observations (each computed series of elements was determined from two days of observations). Figures 13 and 14 show the variation in eccentricity e and perigee longitude ω of the satellite "Echo 1" during the period August 12–25, 1960, since the radiation pressure effect manifests itself most strongly in precisely these elements, whereas for the other elements (a,Ω,i) the effect over such a short period is comparable with observational errors. From the position of the experimental points, marked by crosses, it is evident that the variation of e and ω is nonuniform: toward the end of the twelve-day period the 24-hr variation of e increased from $3\cdot10^{-4}$ to $5\cdot5\cdot10^{-4}$. Each figure shows four curves, corresponding to different hypothetical cases. Thus, e.g., case D presupposes the absence of radiation pressure which leads to considerable discrepancy with the experimental points. In case D the eccentricity decreases due to atmospheric drag and the third harmonic, whereas the perigee longitude increases linearly under the influence of the second harmonic. The cases A, B, and C include radiation-pressure effects, but they correspond to different values of A/m–more exactly, of $\mathfrak{R}(A/m)$. In the case of a sphere of the "Echo 1" type, it is precisely this quantity

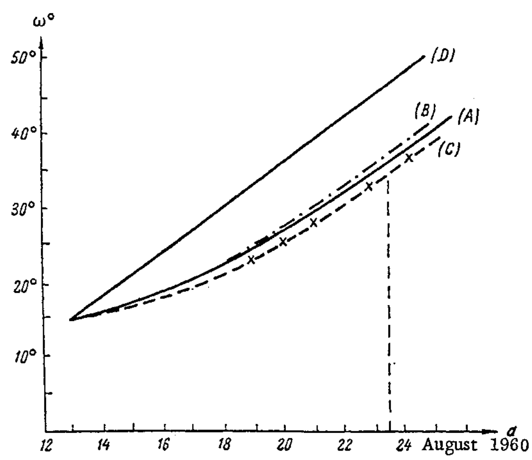


Fig. 14 Perigee-longitude variation of satellite Echo 1 over period August 12–25, 1960 (dashed line denotes time of first entry into Earth's shadow).

which ought to be regarded as a separate parameter, since an exact definition of $\mathfrak{R}$ and A/m is difficult. The coefficient $\mathfrak{R}$ varies from 0 to 2 and depends on the reflectivity and geometrical shape of the surface. For specular reflection from a geometrical sphere, $\mathfrak{R} = 1$, but even small deviations from sphericity or from specular reflection lead to an increase in $\mathfrak{R}$. For completely diffuse reflection, $\mathfrak{R} = 1.44$. Moreover, if the satellite surface oriented toward the sun is strongly heated, then additional infrared reradiation appears which also causes an increase in $\mathfrak{R}$.

The cases A, B, and C correspond to the following extreme values of A/m cm^2/g (Table 5).

In case A, the quantity $\mathfrak{R}(A/m)$ increases by 2% per day from an initial value of 102 cm^2/g for $\mathfrak{R} = 1$. This is possible if one assumes that the chemical which liberates the gas, with a mass of about 15 kg, was decomposed almost entirely during 10 days of satellite life (which involved a mass loss of about 1.5 kg per day), whereupon the mass loss stopped.

Case B presupposes mass invariability for $\mathfrak{R} = 1$ (specular reflection). Case C corresponds to $\mathfrak{R} = 1.13$ with a steady daily loss of mass of about 400 g. From the figures it is evident that the indicated three hypothetical cases differ little from each other, although case C gives best agreement with observation. This is apparently because the reflection coefficient of aluminum is 0.85–0.90 and not unity, as was assumed in the foregoing. For ω, the difference between these three cases and D is very small, which can be readily explained: the line earth-sun approaches the normal to the line of apsides, whereas the radiation pressure tends to shift the orbit in the direction of the line of apsides, as a result of which its effect on ω decreases. On August 24, 1960, the line earth-sun was exactly perpendicular to the line of apsides, so that ω did not vary at all due to the radiation pressure.

The satellite "Echo 1" entered the earth's shadow for the first time on August 23, 1960, on its 140th orbit, approximately; however, as is evident from Figs. 13 and 14, e and ω do not react to the shadow, which we shall regard as a right circular cylinder of radius equal to the mean radius of the earth. In the elements i and Ω only insignificant variations, not exceeding 10^{-4} deg/rev, appear, at first; however, if the shadow is asymmetrical, these variations can become significant; thus, if the shadow is asymmetrical with respect to the line of nodes, the maximum variation occurs in Ω, whereas in the case of asymmetry with respect to the normal to the line of nodes, it occurs in i. A comparison of theory with observations of "Echo 1" also showed that on entry into the earth's shadow the mass loss decreases. The fluctuations of the balloon, i.e., its periodic expansion by heating, is also expected to contribute to the variation in $\mathfrak{R}(A, m)$.

Fig. 15 shows the variation in perigee height h_p and mean altitude h_{mean} during a period of about 900 days from the vernal equinox of 1960, for a constant value of $\mathfrak{R}(A/m) = 102$ cm^2/g. The perigee height has a period of oscillation of about 300 days, with an amplitude of about 500 km. The maximum ordinate is seen to decrease monotonically. The initial variation of h_p was about 2 km per day. Then it increased to 5 km per day, as a result of which h_p decreased by 44 km during the fortnight under consideration. The initial mean altitude h_{mean} was 1600 km; it decreases, but not monotonically, since the radiation pressure causes a change in orbital energy upon entering the shadow. The curve representing the variation of h_p is, however, unsuitable for estimating the lifetime of "Echo 1," which involves the exact determination of density and mass loss. Taking the curve of Fig. 15 only as a tentative basis, and assuming that the size of the satellite will not fundamentally change later on, it can be conjectured that the satellite "Echo 1" will cease to exist in July 1963, at a moment when the perigee height will reach a minimum. However, in view of a weakening in solar activity in the next few years, a longer lifetime could also be expected.

The perturbations of the elements of the satellite "Echo 1," computed by Musen's formulas, are in agreement with the

Table 5

	August 13, 1960	August 23, 1960
(*A*)	102	130
(*B*)	102	102
(*C*)	115	125

experimental data processed by Shaprio and Jones; thus, e.g., the initial variation in e was found to be $2 \cdot 10^{-4}$ per day, and the variation in h_p about 2 km per day.

The fact that e is small, whereas ω varies very rapidly, creates a number of difficulties in perturbation studies. In this case, it is more convenient to determine the perturbations by Kozai's method,[8] by using the elements

$$\xi = e \sin\omega \qquad \eta = e \cos\omega \tag{46a}$$

which he suggests should be determined by the formulas (neglecting the earth's shadow):

$$\xi = \xi' + 0.00066 = 0.04409 \sin(91°99 + 2°965t) + 0.03579 \sin(\lambda' + \Omega) + 0.00188 \sin(\lambda' + \Omega) - 0.00519 \sin\lambda' + 0.00066 \tag{46b}$$

$$\eta = 0.04409 \cos(91°99 + 2°965t) + 0.03373 \cos(\lambda - \Omega) - 0.00200 \cos(\lambda' + \Omega) - 0.00169 \cos\lambda'$$

The term 0.00066 appears in ξ as a consequence of odd harmonics in the expansion of the earth's potential, the coefficient 0.04409 is the unperturbed value of the eccentricity, whereas all the other terms are due to the radiation pressure. The unperturbed value of the eccentricity and the value of the perigee argument at the initial time (which is equal to 90°99) are determined by Kozai from observations, by the method of least squares.

The mean elements of "Echo 1" were also determined by Zadunaisky, et al.,[21] on each day from August 13, 1960 to January 11, 1961, by processing radio and photographic observations distributed over a two-day arc, i.e., over 25 orbits. The first-order short-period perturbations due to the second harmonic were calculated analytically and then subtracted from the observations.

5. Radiation Pressure and Atmospheric Drag

Let us now compare the effects of radiation pressure and of atmospheric drag on satellite motion. It is well known that the instantaneous tangential acceleration of a satellite that moves through a stationary atmosphere is determined by the formula

$$\mathfrak{T} = -\frac{1}{2} c_D \rho v^2 \frac{A}{m} \tag{47}$$

where c_D is the dimensionless aerodynamic drag coefficient, v the orbital velocity of the satellite, ρ the atmospheric density (which depends on altitude and which varies according to the exponential law $\rho = \rho_p \exp(\Delta h/H)$). In this formula ρ_p is the density at perigee height, Δh the difference between satellite and perigee height, and H the scale height.

On the other hand, the perturbing acceleration under the effect of radiation pressure is determined by the (already known) formula

$$F = \mathfrak{P}r(A/m) \tag{48}$$

Their ratio

$$\left|\frac{\mathfrak{T}}{F}\right| = \frac{1}{2} \frac{c_D \rho v^2}{\mathfrak{P}r} \tag{49}$$

does not depend on the characteristics of the satellite. In view of the fact that $c_D = 2$, that the circular velocity of the satellite $v \approx 8$ km/sec, and that the radiation-pressure force on the reflecting surface $\mathfrak{P}_r = 0.9 \cdot 10^{-4}$ g/cm·sec^2, we shall obtain

$$\left|\frac{\mathfrak{T}}{F}\right| \approx 1.2 \cdot 10^{-16} \rho \tag{50}$$

This formula permits us to estimate at once $|\mathfrak{T}/F|$ for any perigee height, if ρ is known sufficiently accurately. Since, for an altitude of 800 km, $\rho \approx 0.8 \cdot 10^{-16}$ g/cm^2 and, hence, $|\mathfrak{T}/F| \approx 1$, it can be asserted that (outside the earth's shadow) the effects of radiation pressure and of atmospheric drag are roughly equal at an altitude of about 800 km for a normal state of the atmosphere (Wyatt,[20] Radziyevskiy and Artem'yev[3]). If the solar activity increases, the upper layers of the atmosphere expand and the level of equal accelerations rises above 800 km. For satellites whose perigee is below this level, e.g., the Soviet artificial satellites, the atmospheric drag is predominant and the radiation-pressure effect can be neglected.

On comparing the radiation-pressure effect and the observed accelerations of the satellite "Vanguard 1" (Jacchia,[6] Briggs[4]), it was found that the variations in satellite orbit are exclusively due to drag, whereas the effect of radiation pressure is small in this case, being at the accuracy limit of period determination. The perigee height of the satellite Vanguard 1 is approximately 650 km. The maximum of $\Delta P/P$, due to drag, was found to be $\pm 0.25 \cdot 10^{-7}$, whereas, due to the radiation pressure, it was 5 to 50 times smaller, depending on the diurnal orientation of the sun with respect to the perigee (if the sun is close to the perigee meridian, the atmosphere expands and the drag increases). In the case of the satellite Vanguard 1, the radiation pressure affects only the perigee height, without affecting the orbital period. The latter can thus be used for determining the structure of the atmosphere, since the observed quantity $\Delta P/P$ permits one to determine the quantity $\rho_p\sqrt{H_p}$, where H_p is the scale height at the perigee height.

The picture differs somewhat for high satellites. It is well known that the satellite "Echo 1" had an initial perigee height of about 1500 km. Since at such an altitude the density $\rho \approx 10^{-18}$ g/cm^3, we estimate $|\mathfrak{T}/F| < 0.01$, so that the relative drag is small. However, as long as the entire orbit is illuminated, the radiation pressure does not affect the period variation (as had been already noted in Sec. 3) and, therefore, one can determine the characteristics of the atmosphere directly from the observed secular acceleration, disregarding the radiation pressure. Only if a high satellite periodically intersects the earth's shadow is it necessary to make allowance for the secular acceleration due to the radiation pressure, by subtracting it from the observed acceleration. However, since the exact determination of the atmospheric characteristics would involve, in this case, the exact determination of the quantity $\mathfrak{R}(A/m)$ (see Sec. 4), the problem becomes too difficult. Only for satellites with pre-assigned $\mathfrak{R}(A/m)$ and small eccentricity of orbit is it possible almost entirely to disregard the effect of radiation pressure on period variation.

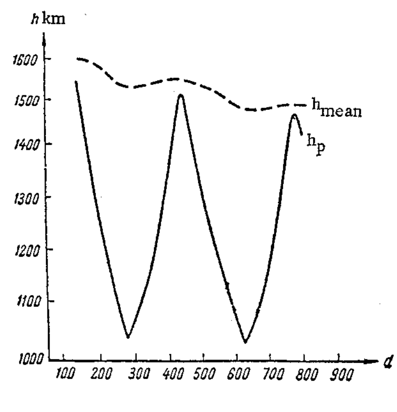


Fig. 15 Variation of mean altitude h_{mean} and of perigee height h_p of satellite Echo 1 (d refers to day of vernal equinox 1960).

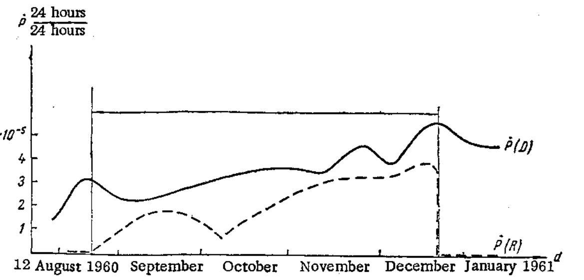


Fig. 16 Rate of increase of orbital period of satellite Echo 1 under the effect of atmospheric drag (solid curve) and of radiation pressure (dashed curve) from August 1960 to January 1961 (arrows denote motion with entry into earth's shadow).

These considerations made it possible to determine the mean density ρ at an altitude of 1500 km (which was found to be $(1.1 \pm 0.2) \cdot (10^{-18}$ g/cm$^3)$ from the orbital-period variation of "Echo 1" from the first few days after launching until it entered the earth's shadow (the variation being +0.0024 min/day). On the basis of the foregoing value for the density, the lifetime of the satellite "Echo 1" was estimated at 20 years, if the radiation pressure is disregarded, and at not more than 1–2 years, if the latter is taken into account (on the assumption that the satellite diminishes in size while retaining its spherical shape). In comparing the theoretical and experimental values of the orbital elements of "Echo 1," Zadunaisky, et al.[21] derived the atmospheric densities for altitudes ranging from 950 to 1500 km over the international ellipsoid on each day from August 13, 1960 to January 11, 1961, for the values $c_D = 2.5$, $H = 200$ km, and $H = 400$ km. For orbits that are partially in the shadow, it is necessary to consider the acquisition or loss of energy by the satellite, since the latter enters and leaves the shadow at different distances from the sun, as a result of which the semimajor axis varies.

Let us now examine the rate of variation of the period of the satellite "Echo 1" due to drag and radiation pressure. In Fig. 16, the curve $\dot{P}(D)$, constructed from Zadunaisky's data, corresponds to the variation in the period P due to drag, and the curve $\dot{P}(R)$ to the variation in the period due to radiation pressure (during the time interval August 12, 1960 to January 1961). During that period, both effects contribute quantities of roughly the same order to $\dot{P}$ (these quantities are comparable even if the perigee height h_p reaches a relative minimum and, hence, the atmospheric density rises considerably). Since the density is altitude dependent, the energy variation due to drag during one orbit, $\dot{E}(D)$ is expressed by a constant term plus even-order terms in e. Analogously, $\dot{E}(R)$ is represented by a linear term plus higher-order terms in e; hence, if the scale height is large enough, $\dot{E}(R)$ increases with e faster than does $\dot{E}(D)$.

As long as the orbit is fully illuminated (i.e., for the first 10 days in August 1960 and the first 14 days in December and in January 1961), the radiation pressure has no effect on the period and $\dot{P}(R) = 0$. The remaining part of the graph corresponds to total and partial shadow: The satellite acquires energy from the sun and then loses it, entering the shadow. Since an illuminated satellite acquires more energy from the sun than it loses as a result of drag, the period increases; this corresponds to the graph's being situated above the t axis.

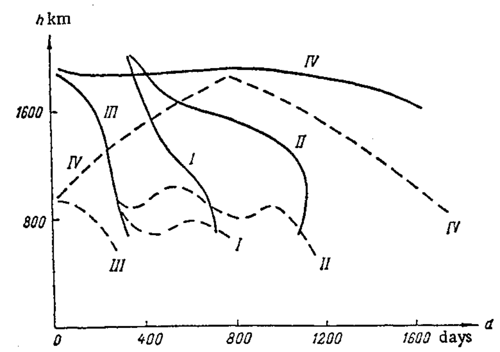


Fig. 17 Variation of mean altitude h_{mean} (solid curves) and of perigee height h_p (dashed curves) of fictitious satellites.

6. Orbital Evolution in Resonance Case and Satellite Lifetime

The effect of resonance on satellite lifetime can be illustrated by the following example: suppose a fictitious satellite with area-to-mass ratio $A/m = 25$ cm^2/g is to be launched into an orbit with a mean altitude of 1700 km and initial elements $a_0 = 1.24\, R_0$, $e_0 = 0.1$, and $\omega_0 = 45°$. By assigning the various values of i_0, Ω_0, and d_0 (the launching day referred to the vernal equinox), it is possible to estimate the satellite lifetime. A calculation, performed by Parkinson et al.[15] for the purpose of estimating the lifetime T, showed that for some resonant combinations the effect of radiation pressure increases, considerably altering the lifetime. This is evident from Table 6, which lists preliminary estimates by Parkinson et al. for the American satellite "Beacon," which was launched in August 1959 but did not enter its orbit.

It can be seen that the lifetime of a satellite in the resonant case can vary by a factor of 10, depending on the time of launching and on the inclination. Considerable variations are also experienced, in this case, by the mean altitude h_{mean} and the perigee height h_p, whose behavior is shown in Fig. 17, where the Roman numerals correspond to the cases of orbital orientation of Table 6. The curves I and II differ only by Ω; they indicate the tentative maximum and minimum of the lifetime when Ω varies, provided the value $i = 48°$ is far enough from the resonance value. The curves III and IV correspond to the case of resonant inclination $i = 40°$, i.e., to the vanishing of the quantity $\dot{\omega} + \dot{\Omega} - \dot{\lambda}'$. In this case, the perigee height either diminishes (III), which gives a short lifetime, or it fluctuates about its initial value (IV), which considerably increases T.

During a full orbit the radiation pressure causes first-order perturbations in all the orbital elements. For circular and almost-circular orbits, the most marked effect is the displacement of their geometric center in the orbital plane along a line which is perpendicular to the line earth-sun, so that the altitude diminishes in that part of the orbit in which the satellite moves away from the sun. Thus, the mean orbital radius remains unchanged. Figure 18 illustrates the continuous action of solar radiation on an initially circular orbit. At point A the satellite moves away from the sun and the radiation pressure raises its velocity and orbital energy, which leads to an increase in the semimajor axis. As a result, the orbit tends to assume the position denoted by the dashed curve passing through point A. At point B the radiation pressure reduces the velocity and energy of the satellite and the semimajor axis decreases. The corresponding orbital position is denoted by the dashed curve passing through B. As a result, the orbit is shifted in the direction indicated by the arrow, i.e., normal to the direction of the radiation-pressure force. Let us denote by $c(\theta)$ the magnitude of this displacement as a function of the angle between the line earth-sun and the orbital plane. The formula, determining the rate of this displacement with allowance for the shadow effect, has the form[15]

$$|\dot{c}(\theta)| \approx \mathfrak{P}_r \frac{A}{m} \left[\frac{3(2\pi - \alpha) + \sin\alpha}{4\pi n}\right] \cos\theta \qquad (51)$$

where α is the angle at the earth's center subtending the orbital arc which is in the shadow, n is the mean diurnal satellite-motion, $\mathfrak{P}_r$ the radiation-pressure force, and A/m the area-to-mass ratio. In comparing $\dot{c}(\theta)$ with the perturbations due to the earth's oblateness, it was found that the altitude of circular orbits fluctuates with a period $2\pi/(\dot{\omega} + \dot{\Omega} - \dot{\lambda}')$ and an amplitude

$$\Delta h \approx |c(\theta)|/(\dot{\omega} + \dot{\Omega} - \dot{\lambda}') \tag{52}$$

Hence, for nearly circular equatorial orbits we obtain

$$\Delta h \approx |c(\theta = 0)|/(\dot{\omega} + \dot{\Omega} - \dot{\lambda}') \tag{52a}$$

Equations (52) and (52a) yield good agreement with observation; thus, e.g., curves I and II of Fig. 17 satisfy Eq. (52a) to within 25%, notwithstanding the effect of drag at low altitude.

In the resonant case, an almost constant orientation of the perigee with respect to the projection of the line earth-sun on the orbital plane is observed, this being due to the earth's oblateness. At the same time, the radiation pressure either reduces or increases the eccentricity monotonically, as a result of which the shape of the orbit can vary considerably. Curves III and IV correspond precisely to this case, the rate of orbital displacement for these curves being directed along the semimajor axis. The lifetime T for an initially circular equatorial orbit with a resonance height of 6000 km is estimated at 1.3 years, and for a height of 1600 km—at a considerably longer period; the amplitude of perigee-height fluctuation of such an orbit is about 100 km.

At first glance, the radiation pressure seems to cause a monotonic decrease in perigee height until the satellite enters the dense layers of the atmosphere and ceases to exist. But, in fact, the orbital behavior of a satellite is influenced by two factors. First, the earth's revolution about the sun causes periodic changes in the direction of solar radiation. Each half-year the radiation pressure shifts the orbit in a direction opposite to the previous, which leads to periodic perigee-height fluctuations. Second, the earth's oblateness forces the orbit to rotate in its plane, and since the radiation pressure shifts the orbit simultaneously in a direction normal to the line earth-sun, the result is again a fluctuating motion of the perigee height. If the position of the perigee is turned by 180°, under the influence of the earth's oblateness, then the perigee height will increase, and not decrease as previously. The forementioned orbits (equatorial and with inclinations not exceeding 50°; Shapiro and Jones[18] call them resonant orbits of the first kind) experience both these effects, and the rotation in the orbital plane due to the earth's oblateness compensates for the change in the direction of solar radiation due to the earth's revolution.

Let us now consider another resonance type for polar- and almost-polar orbits, called resonance of the second kind.[16] For large inclinations, comprising also the case of retrograde motion, the orbital evolution becomes fairly complicated. The resonance condition is, in this case, the vanishing of the quantity $\dot{\omega} - \dot{\Omega} + \dot{\lambda}'$. As an example, we can consider a satellite launched into a resonant polar orbit at dawn on the day of winter solstice. Since the orbit is polar, $\Omega = 0$ and $\dot{\omega} < 0$, and the radiation-pressure force component which lies in the orbital plane causes the perigee to approach its position at the moment of launching. This component rotates in the orbital plane with the same velocity and in the same direction as the position of the perigee due to the earth's oblateness, i.e., it tends to 90°. On the days of vernal and autumnal equinoxes the vector of solar-radiation pressure lies entirely in the orbital plane and the perigee height decreases with maximum rate.

If such a satellite is launched at dusk, then the radiation-pressure force component and the perigee position move in different directions, which considerably increases the satellite lifetime. It is evident that the time of launching is also of decisive importance in the orbital evolution in the case of orbits with large inclinations.

In the work of Parkinson, et al., the following estimates are made for the satellite "Echo 1" ($A/m \approx 125$ cm²/g): if the

Table 6 Satellite lifetime in resonance and nonresonance case

	i_0	d_0	Ω_0	T(days)	
I	48°	150	300°	340	no resonance
II	48	150	120	760	
III	40	0	45	180	resonance
IV	40	0	225	1700	

satellite is launched into a circular orbit with a mean altitude of 1600 km and a moderate inclination $i \approx 35°$, then the life time is estimated at only 240 days, since the inclination is close to its resonance value $i = 40°$. If such a satellite is launched into a polar circular orbit with an altitude of 4000 km, then the lifetime varies between 1.3 to 3 years, depending on the launching time.

For polar- and almost-polar orbits whose direction of rotation due to the earth's oblateness is opposite to the direction of rotation of the line earth-sun, resonance of the second kind gives (as a rule) small fluctuations in the general course of perigee-height decrease.

Resonance of the second kind is obtained from resonance of the first kind by replacing direct satellite motion by retrograde motion, i.e., by replacing i with $180°-i$. The greatest discrepancy between both types of resonance is obtained for $\Omega = 0°$ (direct motion), and $\Omega = 180°$ (retrograde motion). For $\Omega = 90°$, the distinction between both resonance types disappears.

The launching of thin dipole-needles into an orbit around the earth (for the purpose of establishing reliable long-range radio communications) can serve as an example of motion along a resonance orbit of the second kind. According to the project "West-Ford," plans were made to launch about 35 kg of thin copper dipoles (which reflect passively radiowaves of wavelength $\lambda = 4$ cm) into a circular polar orbit with a mean altitude of 3800 km. It was assumed that each dipole is 1.77 cm long and 0.000286 cm in diameter, the container being filled with $3.5\ 10^8$ dipoles which were to be scattered along the orbit, creating (approximately 2 months after launching) a belt around the earth 8 km wide and 40 km thick (radially) with a density of 21 dipole/km³. The possible effects of such a belt on radio communications, radar, optical- and radio-astronomical observations were dealt with in detail in a series of papers published in the Astronomical Journal **66**, No. 3 (1961).

At an altitude of 3800 km, the atmospheric pressure is negligible, and the belt can be gradually lowered into the lower layers of the atmosphere by using the radiation-pressure force, provided that the dipoles are launched into an appropriate resonance orbit.

Suppose the dipoles, with an area-to-mass ratio $A/m \approx 50$ cm²/g, are launched into a resonance orbit. The mean orbital altitude (3800 km) is selected, a priori, such that the radiation-pressure force would increase the eccentricity and lower the perigee in the atmosphere, without altering the structure of the belt. Under such initial conditions the mean lifetime of the belt is 7 years, after which the dipoles scatter in space. The lifetime useful for radiocommunications is about 2 years. We shall consider, as hitherto, the component of the radiation-

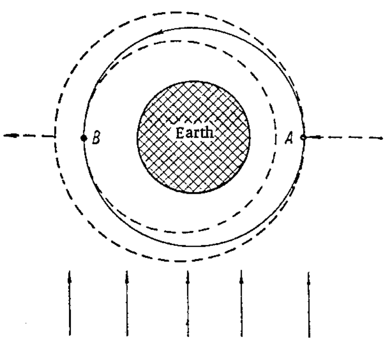


Fig. 18 Evolution of initial circular orbit under effect of radiation pressure (direction of orbital shift is denoted by dashed arrows).

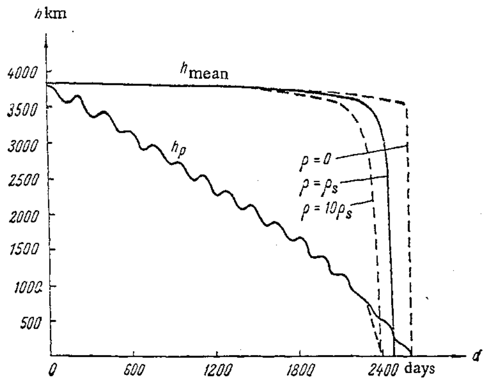


Fig. 19 Variation of mean altitude h_{mean} and of perigee height h_p of West-Ford type of resonance orbit.

pressure force which lies in the orbital plane, assuming that all the other components are small. Since the orbit is polar, it follows that

$$\dot{\Omega} = 0, \dot{\omega} = \frac{1}{2} J \frac{n}{a^2(1 - e^2)^2}$$

and the resonance conditions are determined by the formula $\dot{\omega} - \dot{\Omega} + \dot{\lambda}' = 0$ or $\dot{\omega} = -\dot{\lambda}'$. Computation shows that the only value of the mean altitude which satisfies this condition is 3800 km. Here, one obtains resonance of the second kind, and the perigee height of the resonance orbit will decrease monotonically, experiencing only weak fluctuations during the general decrease. The mean altitude of the orbit decreases, in this case, entirely irregularly (Fig. 19).

In addition to the exact resonance orbit, there exists a bundle of resonance orbits which are close to the mean altitude of 3800 km; the thickness of this bundle is directly proportional to A/m and for $A/m \approx 50$ cm²/g it is equal to 300 km. For orbits which lie outside the resonance bundle, $|\dot{\omega}|$ will differ markedly from $|\dot{\lambda}|'$, as a result of which the effect of orbital rotation due to the earth's oblateness, and the effect of the radiation pressure will not compensate each other, thus resulting in periodic fluctuations of the perigee height.

The lifetime of dipoles in a resonance orbit depends on the initial orbital altitude, the inclination, and the area-to-mass ratio, but depends only slightly on atmospheric drag. Figure 19 illustrates this for resonance of the second kind. Three models of the atmosphere were selected: 1) $\rho = \rho_s$, determined from observations of the orbit of the satellite "Echo 1"; 2) $\rho = 10\rho_s$; 3) $\rho = 0$. From the figure, it is evident that each model of the atmosphere yields a difference of not more than 10% in the determination of the lifetime of the dipole belt. Only for high orbits which are far from resonance, and which consequently have long-period fluctuations in h_p, does the lifetime depend on ρ. In this latter case, the effect of the atmosphere on the mean-altitude variation during one orbit is determined by the formula

$$\Delta h_{mean} = -2\pi c_D (A/m) a^2 \rho \tag{53}$$

The results of lifetime calculations of a dipole belt T in a resonance and nonresonance orbit, respectively, are shown in Fig. 20. This computation was performed by Shapiro and Jones[18] for a dipole belt with $A/m \approx 50$ cm²/g on the assumption that the launching into an orbit with elements $e_0 = 0$, $\Omega_0 = 10°$ takes place at dawn in a southerly direction. It is precisely in this case that resonance of the second kind maximizes T. Launching in the evening into a resonance orbit of the first kind gives a value of T smaller by a few years. In Fig. 20, T is expressed by the numericals inside the contours. The orbits which lie outside the resonance region, i.e., outside the contour, are subject to the influence of the atmosphere and are rapidly leaving the resonance region. The dipole belt under consideration will have an orbit which lies inside an 8-year contour and will have a lifetime of about 7 years.

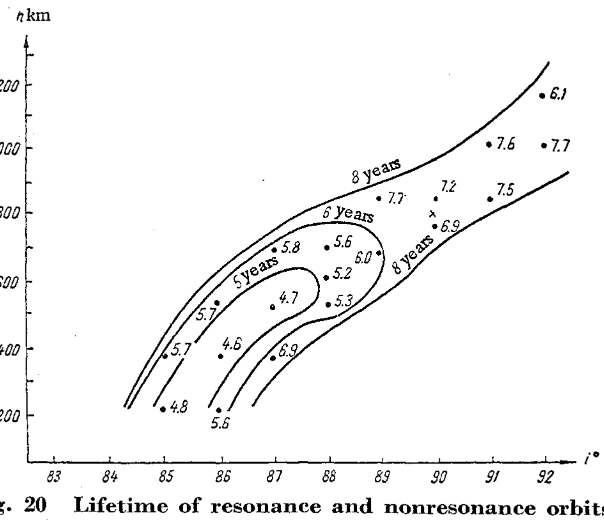


Fig. 20 Lifetime of resonance and nonresonance orbits (the cross denotes a West-Ford type of orbit).

The object of the present paper is a brief review of current literature on the effect of radiation pressure on the motion of artificial earth satellites. Even at present, the perturbations due to radiation pressure are of major importance for some existing satellites (such as "Echo 1"). In the course of future space investigations and the launching of new artificial bodies, the range of problems related to the study of, and accounting for, the effect of radiation pressure will unavoidably increase. Therefore, a thorough investigation of the effects of radiation pressure on the motion of artificial bodies becomes more and more necessary.

In conclusion, we express our gratitude to Yu. V. Baratkov, who reviewed the manuscript and made a number of remarks.

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Some Modficiations in Method of Improving the Orbits of Artificial Earth Satellites

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A method involving the use of a rotating system of coordinates is proposed for improving the orbital elements of artificial satellites.

THE problem of orbit improvement amounts to the determination of the system of elements that best fits the observations. The orbits of artificial satellites can be improved by using any of the classical methods of orbit correction developed for the study of the motion of celestial bodies.

Let us use the method set forth in M. F. Subbotin's *Course in Celestial Mechanics* (1941) under the designation of "second method." The basic formulas that determine the position of a satellite (α,δ) on the celestial sphere are

$$\begin{aligned} x - X &= \rho \cos\delta \cos\alpha \\ y - Y &= \rho \cos\delta \sin\alpha \\ z - Z &= \rho \sin\delta \\ \rho^2 &= (x - X)^2 + (y - Y)^2 + (z - Z)^2 \end{aligned} \tag{1}$$

where ρ is the distance from the observer to the satellite, x,y,z are the rectangular geocentric coordinates of the satellite, and X,Y,Z are the coordinates of the observing station. By differentiating and transforming formulas (1), we obtain the equations of condition connecting the rectangular-coordinate corrections with the deviations of the calculated positions of the body (α_c,δ_c) from the observed positions (α_0,δ_0):

$$\begin{aligned} \rho \cos\delta d\alpha &= -\sin\alpha dx + \cos\alpha dy \\ \rho d\delta &= -\sin\delta \cos\alpha dx - \sin\delta \sin\alpha dy + \cos\delta dz \end{aligned} \tag{2}$$

where dx,dy, and dz are expressed in terms of the partial derivatives of the rectangular coordinates x,y,z with respect to the orbital elements, as follows:

$$\begin{aligned} dx &= \frac{\partial x}{\partial \Omega} d\Omega + \frac{\partial x}{\partial i} di + \frac{dx}{\partial M_0} dM_0 + \frac{\partial x}{d\omega} d\omega + \frac{\partial x}{\partial \varphi} d\varphi + \frac{\partial x}{\partial n} dn \\ dy &= \frac{\partial y}{\partial \Omega} d\Omega + \frac{\partial y}{\partial i} di + \frac{\partial y}{\partial M_0} dM_0 + \frac{\partial y}{\partial \omega} d\omega + \frac{\partial y}{\partial \varphi} d\varphi + \frac{\partial y}{\partial n} dn \\ dz &= \frac{\partial z}{\partial \Omega} d\Omega + \frac{\partial z}{\partial i} di + \frac{\partial z}{\partial M_0} dM_0 + \frac{\partial z}{\partial \omega} d\omega + \frac{\partial z}{\partial \varphi} d\varphi + \frac{\partial z}{\partial n} dn \end{aligned} \tag{3}$$

where Ω is the longitude of the ascending node from the point of vernal equinox, i the orbit's inclination to the equator, M_0 the mean anomaly at time t_0, ω the argument of perigee, φ the angle of eccentricity (related to the eccentricity by the formula $e = \sin\varphi$), and n the diurnal motion of the satellite. The correction of the orbital elements reduces to the calculation of these partial derivatives and to the determination of the corrections of the elements $d\Omega$, di, etc., by the method of least squares.

The special feature of the "second method" consists in the fact that the rectangular coordinates of the celestial body x,y,z, as well as the corresponding formulas for $\dot{x},\dot{y},\dot{z}$, are expressed in terms of the *true* anomaly by the formulas:

$$\begin{aligned} x &= r(\cos u \cos\Omega - \sin u \sin\Omega \cos i) \\ y &= r(\cos u \sin\Omega + \sin u \cos\Omega \cos i) \\ z &= r \sin u \cos i \\ u &= v + \omega \\ r &= \frac{\bar{a} \cos^2\varphi}{1 + \sin\varphi \cos v} \end{aligned} \tag{4}$$

or

$$\begin{aligned} x &= r \sin a \sin(A + u) \\ y &= r \sin b \sin(B + u) \\ z &= r \sin c \sin(C + u) \end{aligned} \tag{5}$$

$$\begin{aligned} \dot{x} &= n\bar{a}[\sin a \cos(A + u)\sec\varphi + \sin a \cos(A + \omega) \tan\varphi] \\ \dot{y} &= n\bar{a}[\sin b \cos(B + u)\sec\varphi + \sin b \cos(B + \omega) \tan\varphi] \\ \dot{z} &= n\bar{a}[\sin c \cos(C + u)\sec\varphi + \sin c \cos(C + \omega) \tan\varphi] \end{aligned} \tag{6}$$

where $\bar{a}$ is the semimajor axis of the orbit.

Using formulas (5) and (6) we obtain the expression for the partial derivatives in terms of the true anomaly. These formulas can be found in M. F. Subbotin's *Course in Celestial Mechanics*.[4]

However, it is rather inconvenient to use these formulas in practice owing to their unwieldiness. The formulas

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